Beta Spectra for ^{39}Ar, ^{85}Kr, and ^{210}Bi

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Abstract

The beta spectra for ^{39}Ar, ^{85}Kr,and ^{210}Bi have been calculated using the theoretical formula given in [1]. The calculated spectra have been compared with those obtained in the low Z limit given by [1, 2]. The spectra have been calculated with and without electron screening effects as proposed by [3, 4]. Finally, tabulated spectra given in [5] where coded and compared to the calculated spectra. The results of these spectra are shown and an overall conclusion of the correct spectral form deduced from a theoretical standpoint. An experimentally determined correction factor [7] is used in obtaining the spectrum for ^{210}Bi.

1 Introduction

There is a clear distinction in the spectrum calculated for allowed and forbidden beta decays. There are three specific corrections considered in this paper: Fermi correction, electron screening, and forbiddeness of the decay. Fermi's correction to the statistical shape of the spectrum takes into account the finite extent of the nucleus. For β^- decays this correction increases the number of low energy electrons in the beta spectrum. The electron screening correction provides a contribution to the low energy portion of the spectrum and is more prominent in large Z atoms. The last correction to the spectral shape is due to the forbiddeness of the decay. For decays that have a difference in spin between the parent and daughter nucleus of larger than one angular momentum, the lepton must carry away the access angular momentum. The forbiddeness correction depends both on the parity change and the angular momentum difference. This correction will provide more high energy electrons in the beta spectrum.

2 Theory of Beta Decay

In our calculation of the beta spectra, we followed [1, 3, 6] in detail for the programming. We have omitted all constants that are not weighting factors in the sum over forbidden states. This was done because the spectra are all normalized to one for comparison. In this formulation the mass of the neutrino is assumed to be zero. This allows us to write $p_\nu = W_\nu$ and the integral over the neutrino momenta can be performed. Thus, using energy conservation we can express all neutrino momenta in terms of the electron energies. Using this approximation and treating the nucleus as a point charge, the general form for the beta spectra can be represented as

$$d\lambda = (W_0 - W)^2 p_e W F(\pm Z, W) S(\pm Z, W) dW. \tag{1}$$

W is the total energy of the electron in units of $m_e c^2$, W_0 is the end-point energy of the electron, and p_e is the electron momentum in units of $m_e c$. The two dimensionless terms $F(\pm Z, W)$ and $S(\pm Z, W)$ are the Fermi correction and the correction due to the forbiddeness of the decay, respectively.

The explicit form of these functions are

$$F(\pm Z, W) = (1+\gamma_0) e^{\pi\nu} \left(\frac{2 p_e R}{\hbar}\right)^{2(\gamma_0 - 1)} \frac{|\Gamma(\gamma_0 + i\nu)|^2}{[\Gamma(2\gamma_0 + 1)]^2} \tag{2}$$

and

$$S(\pm Z, W) = \sum_j \frac{(2j)!!}{(2J-2j)!!} \frac{L_{j-\frac{1}{2}} (W_0 - W)^{2(J-j)-1}}{(j-\frac{1}{2})!(J-j-\frac{1}{2})!} M_j^2(j) \tag{3}$$

where $\gamma_0 = [1-(\alpha Z)^2]$, $\nu = \pm\frac{\alpha Z W}{p_e}$, $R = \frac{1}{2}\alpha A^{\frac{1}{3}}$ in units of $\frac{\hbar}{mc}$. Z is the number of protons in the daughter nucleus, A is the number of nucleons, and α is the fine structure constant. The sum is performed over $j = \frac{1}{2}, .., \Delta J - \frac{1}{2}$ where ΔJ is the difference in total angular momentum of the parent and daughter nuclei. The shape factor, $S(\pm Z, W)$, is expressed using an approximation, in which only the lowest order terms in R of the expansion of the nuclear matrix elements are retained.

Inside the sum we have the term $M_j^2(j)$ which is some linear combination of the appropriate nuclear matrix elements. The final contribution to the sum is from the term

$$L_{j-\frac{1}{2}} = \frac{1}{2}(2p_e)^{2(\gamma-\gamma_0)} \left[\frac{\Gamma(2\gamma_0+1)}{\Gamma(2\gamma+1)}\right]^2 \left|\frac{\Gamma(\gamma+i\nu)}{\Gamma(\gamma_0+i\nu)}\right|^2 R^{2(\gamma+\gamma_0)-(2j-1)} k(k+\gamma) \tag{4}$$

where $\gamma = [(j+\frac{1}{2})^2 - (\alpha Z)^2]^{\frac{1}{2}}$ and $k = j + \frac{1}{2}$.

To apply an electron screening correction one needs to multiply the spectrum by the dimensionless factor

$$\left[\frac{(W \mp V_0)^2 - 1)}{W^2 - 1}^{\frac{1}{2}}\right] \left(\frac{W \mp V_0}{W}\right) \tag{5}$$

where $V_0 \approx 1.13\alpha^2 Z^{\frac{4}{3}}$. When applying this correction, the Fermi function should also be corrected such that $F(\pm Z, W)$ has the potential correction $F(\pm Z, W \mp V_0)$. $S(\pm Z, W)$ is a correction applied due to the ability, or inability, of the atom to decay with the needed angular momenta and therefore no screening correction is needed.

3 First Unique Forbidden Decays

3.1 Determination of Correct Spectral Equation

In looking for the correct form of the shape factor $S(\pm Z, W)$, we start with the reasoning given in [1, 6]. Konopinski argues [1] that for the unique first forbidden decays the matrix elements $M_j^2(j)$ found in the sum, are nearly constant and can be brought outside the sum. Thus, we have left this contribution out of our calculated beta spectrum since it is taken into account only as an overall constant. In the determination of the correct form of $d\lambda$ for the first unique forbidden decays, we have drawn our conclusion based on the spectral form of the tabulated values given in [5] since there was no available experimental data.

In comparing the two spectral forms given by [1, 6] and [5], it was found that they did not match initially. To determine the cause for this inconsistency we looked specifically at the terms in the Fermi correction and shape factor and compared them to the tabulated data. Since our program produces 1 keV binning, our spectrum is smooth in comparison to what is obtained in the tables. In order to obtain a comparable curve from the tabulated values, which are given in p_e, we performed a linear interpolation of the data. Since the λ values are not very linear at low energy, the spectrum becomes smooth after $p_e \approx 0.5$ or 40 keV. It can be seen in Figure 2 that the calculated Fermi corrections as a function of p_e matches the tabulated data well for momenta greater than 0.3.

To determine the correction for the shape factor, the sum in Eq. (3) was initially expanded in the low Z approximation proposed by [1] and [2]. This gives a shape factor $S(W, Z) = p_e^2 + (W_0 - W)^2$. However, this form does not give the same spectrum as seen in the tables. This deviation is due to the approximation $\alpha Z << 1$ not being valid. Konopinski does suggest [1] a modification to this form such that $p_e^2 + (W_0 - W)^2 \longrightarrow p_e^2 + (W_0 - W)^2 + 9\frac{L_1}{L_0}$. The added terms are tabulated values of the electron wave-functions. This correction is explained only as an effect due to large Z. Due to this explanation, the explicit form of the sum found in [1] was used. Since the shape factor in [5] is given as

$$C(W) = \frac{R^2}{9}({}^A F_{211}^{(0)})^2 (p_\nu^2 + \lambda_2 p_e^2) \tag{6}$$

the expression given by $S(\pm Z, W)$ was expanded to find the momentum dependent term λ_2. This can be done by expanding the sum for $\Delta J = 2$ which gives two terms, $j = \frac{1}{2}, \frac{3}{2}$.

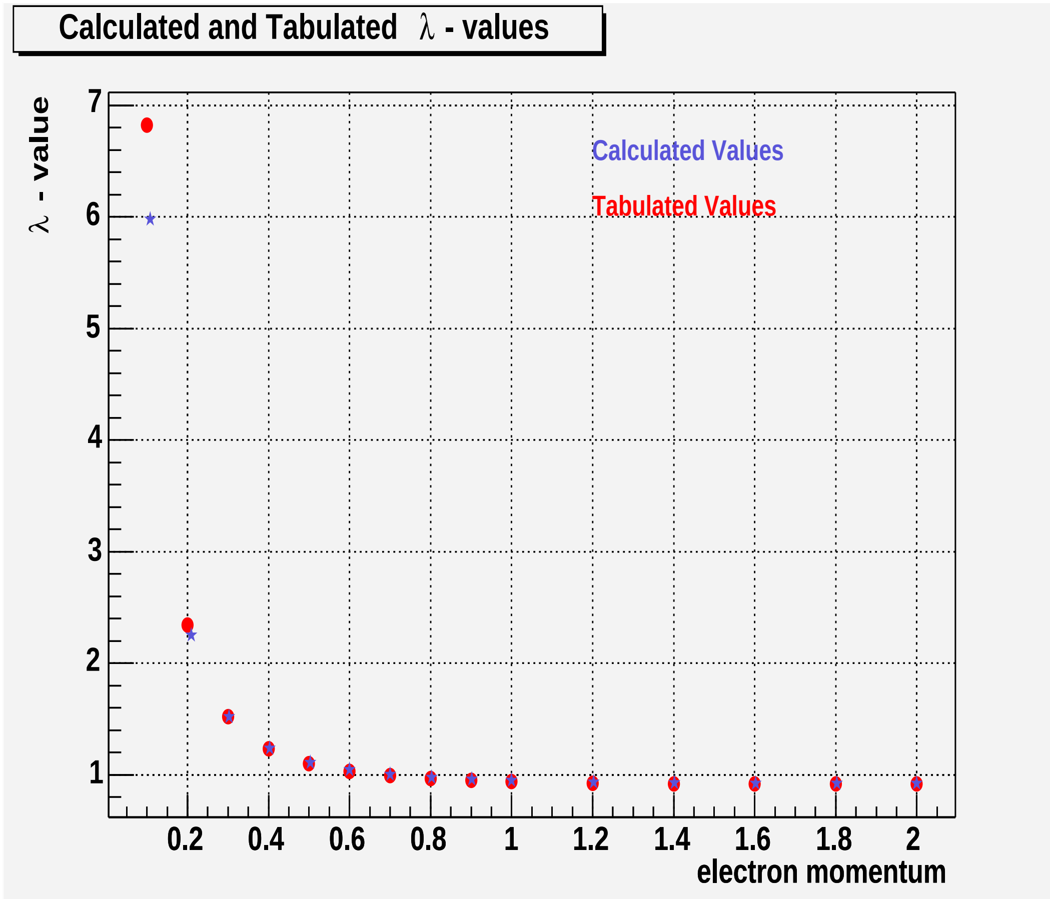

Figure 1: ^{85}Kr λ values.

Separating terms and performing some algebra one can show that

$$\lambda_2 = 72 \frac{\gamma + 2}{\gamma_0 + 1} (2pR)^{2(\gamma - \gamma_0 - 1)} \left[\frac{\Gamma(2\gamma_0 + 1)}{\Gamma(2\gamma + 1)} \right]^2 \left| \frac{\Gamma(\gamma + i\nu)}{\Gamma(\gamma_0 + i\nu)} \right|^2. \tag{7}$$

The values for λ_2 are comparable for $p_e > 0.2$, which corresponds to an electron kinetic energy of $\approx$ 10 keV. It can be verified that this is the exact same form as given by Eq. (3) in [4].

3.2 ^{85}Kr Spectrum

There is a problem in the above theory for the electron momentum if one considers electron screening at low energies. This is due to the momentum being represented as $p_e =$

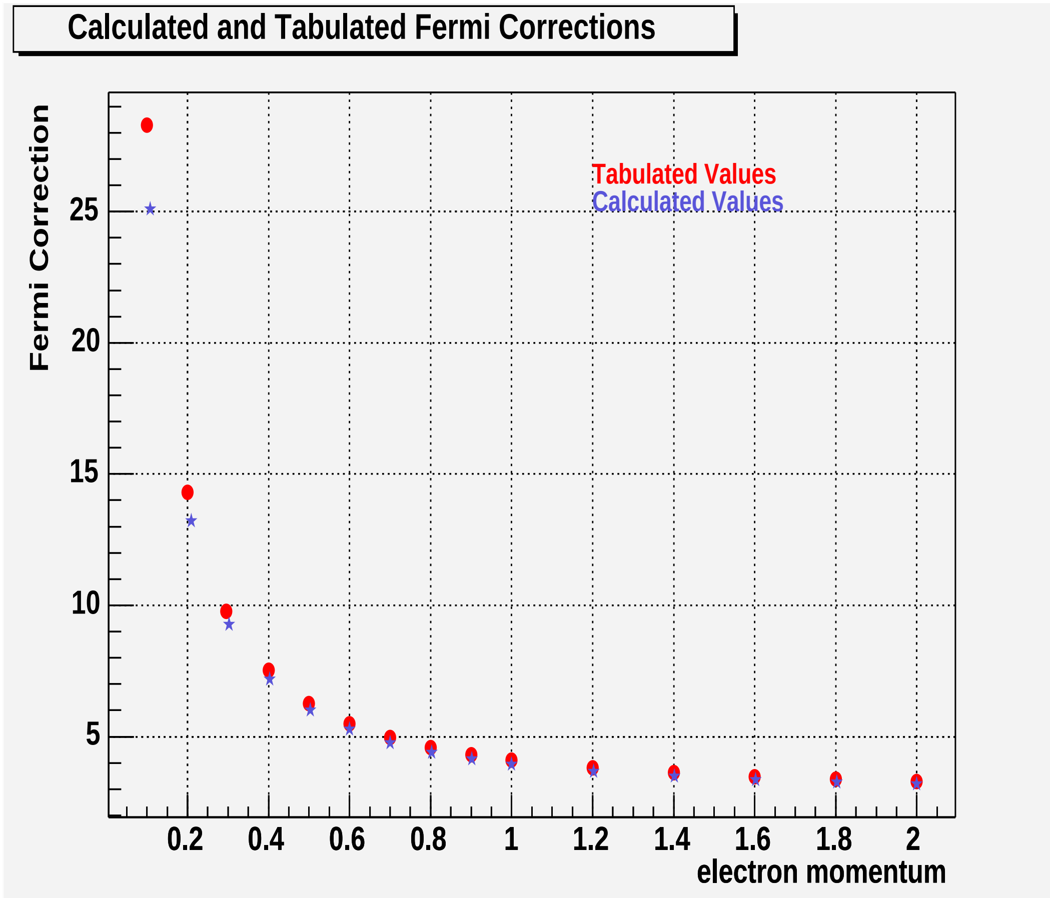

Figure 2: Comparison of ^{85}Kr values for the Fermi Correction.

$\sqrt{(W - V_0)^2 - 1}$ in which there are now possible values for the electron kinetic energy, which lead to imaginary momenta. In our code we fixed this by making $V_0 = 0$ when this occurs, but obviously this is not justified theoretically. However, since we are not concerned with energies this low in the KamLAND data we have chosen to only consider the beta spectrum above 80 keV. Although we believe the calculated values are good to within 20 keV, cutting the spectrum here results in less than a 1.0 % deviation in the calculated values, with respect to the tabulated values, and is more than sufficient for our purposes.

In looking at the possible beta spectra seen in Figure 3 there is a large difference between the purely statistical shape and the corrected shapes. When all of these corrections are applied to the statistical shape, at higher energies (e^- kinetic > 450 keV) the spectra are comparable to the purely statistical shape. However, the need to apply all these corrections is seen in the

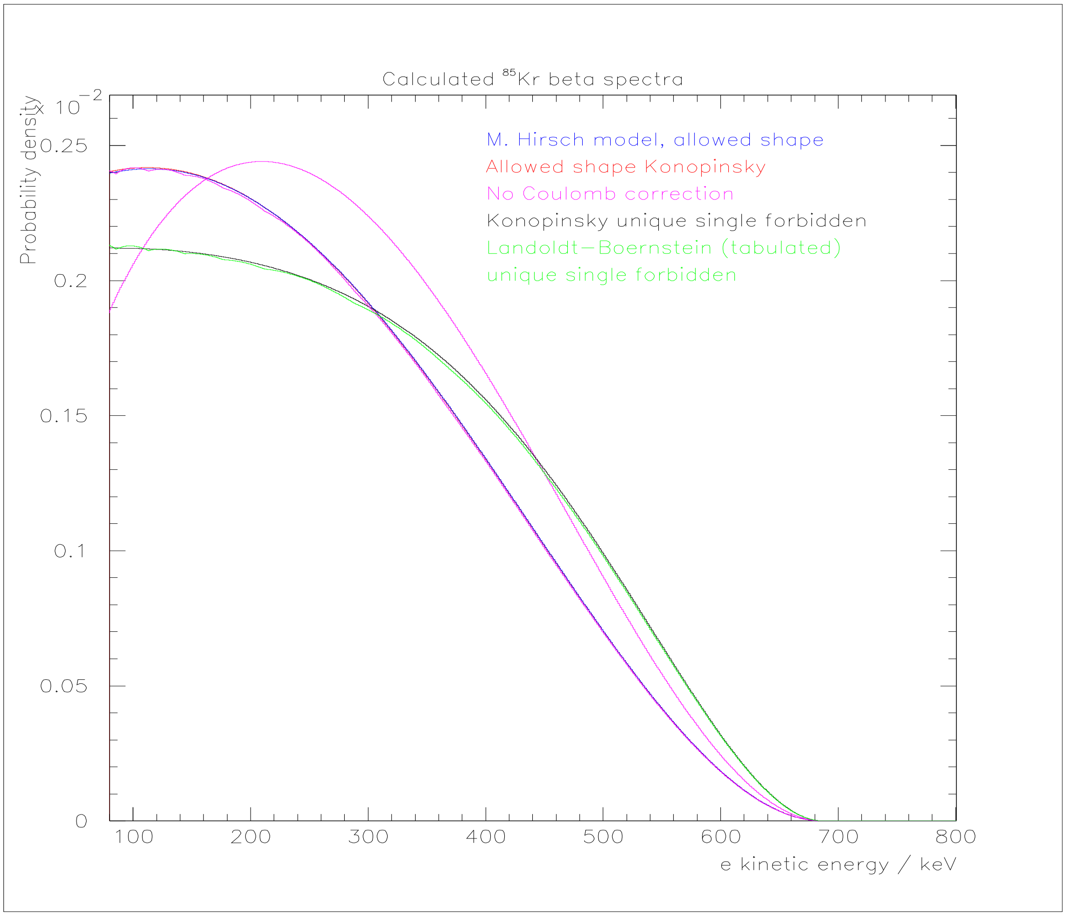

Figure 3: Calculated and tabulated beta spectra for ^{85}Kr.

large deviation from the statistical shape at energies lower than 450 keV. The total correction needed for the statistical shape can be seen in Figure 4. The last possible correction one can apply to the statistical shape would be for electron screening. This was done for ^{85}Kr to see what effect this correction has on the calculated first unique forbidden spectrum. The correction factor can be seen in Figure 5. This is less than a 1 % correction. Since ^{85}Kr should have the largest correction factor to the first unique forbidden spectrum due to it having a greater charge than does ^{39}Ar, we can conclude that this correction is negligible with respect to the Fermi and shape factor corrections.

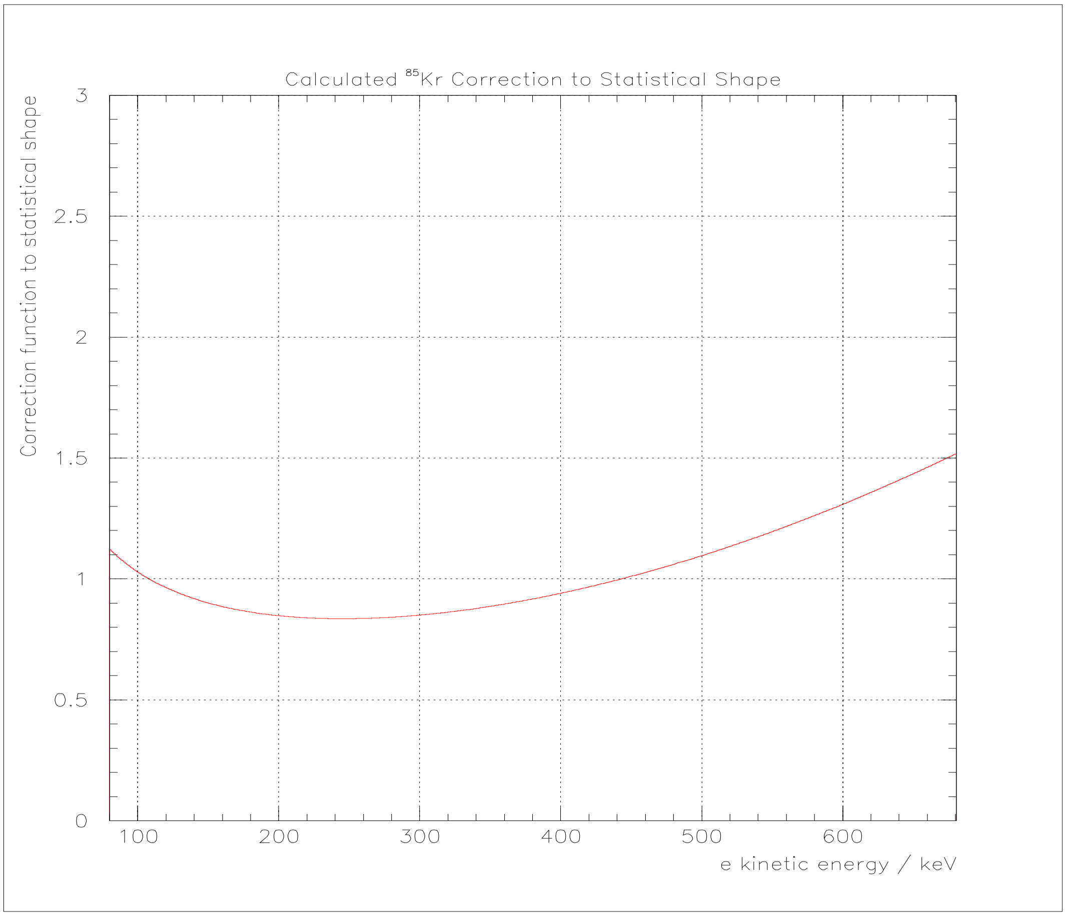

Figure 4: ^{85}Kr correction to statistical shape.

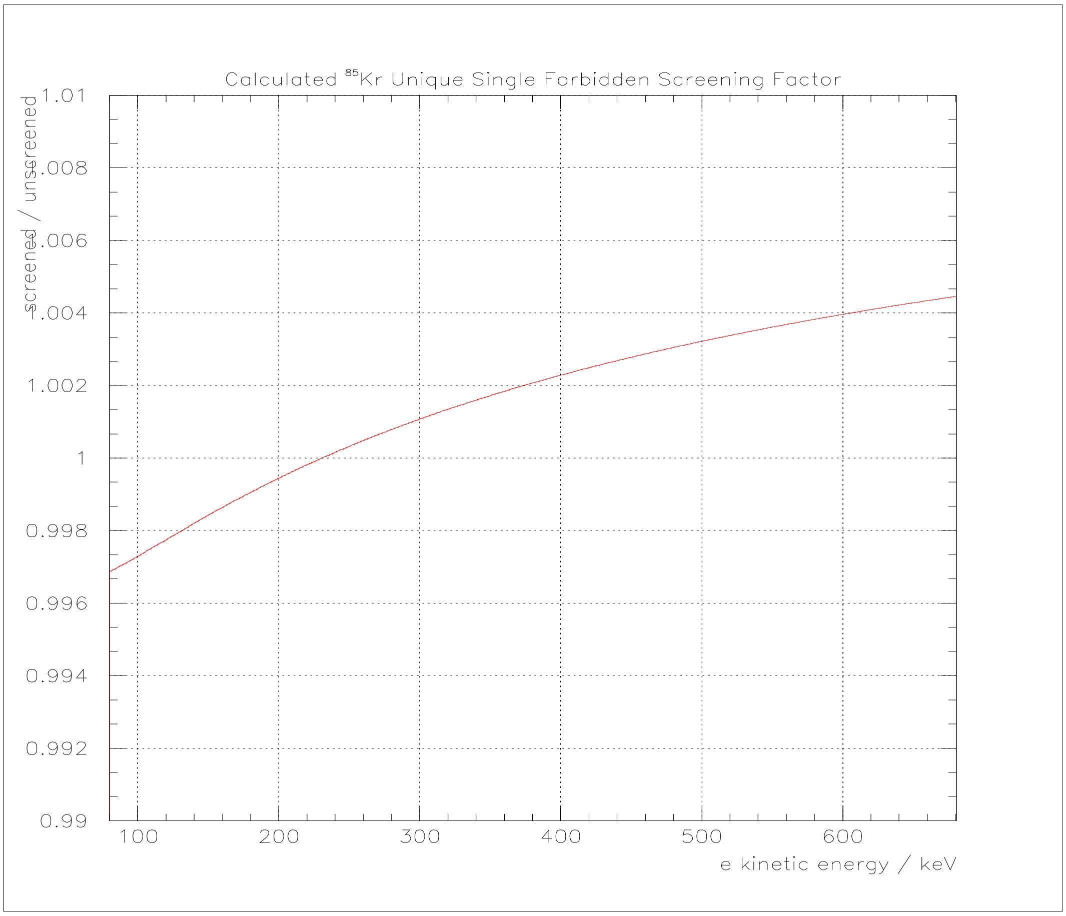

Figure 5: ^{85}Kr screening correction to unique first forbidden shape.

3.3 ^{39}Ar Spectrum

The same analysis described for ^{85}Kr has been performed for ^{39}Ar. Figures 6 and 7 show the values we calculated and those obtained in the tables [5]. There is again a less than 1.0% deviation in the ratio of calculated to tabulated values. Figure 8 shows the statistical,

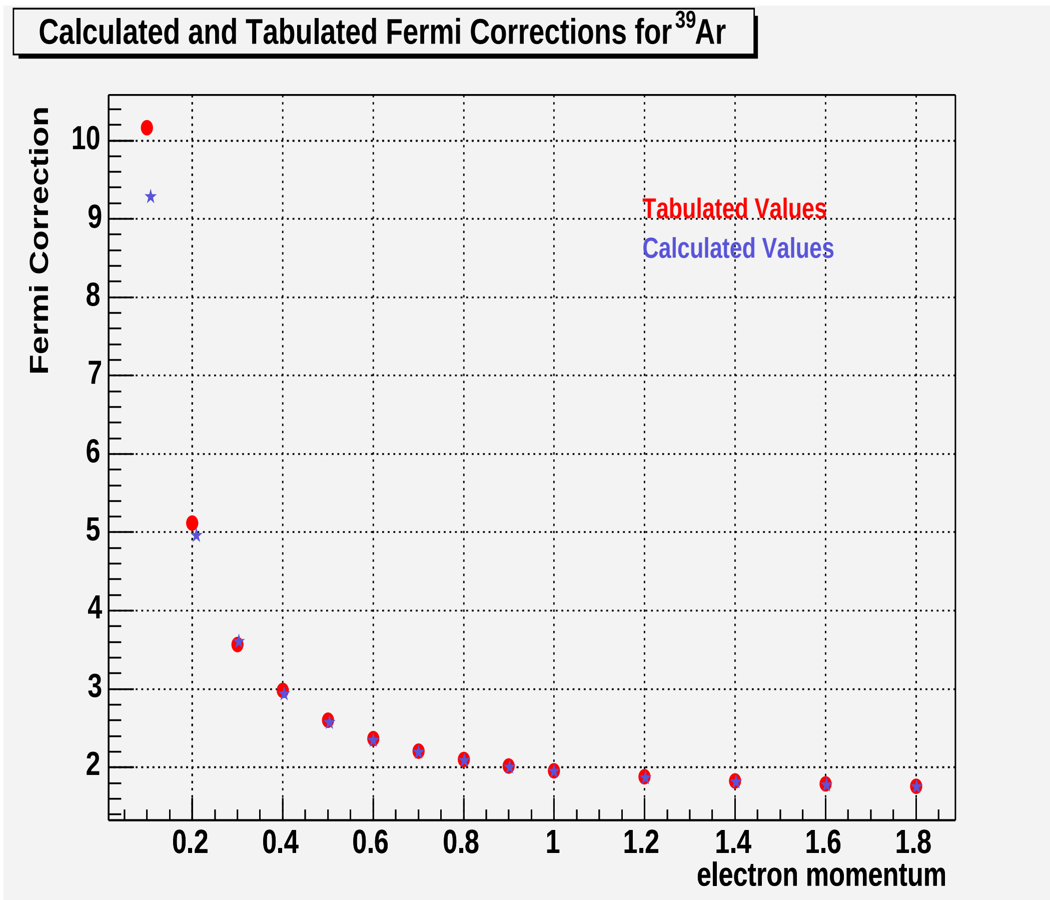

Figure 6: Comparison of ^{39}Ar values for the Fermi correction.

allowed, and forbidden spectra for ^{39}Ar. The total correction factor needed to be applied to the statistical shape is given by Figure 9. There is good agreement between our calculated spectra and the tabulated spectra.

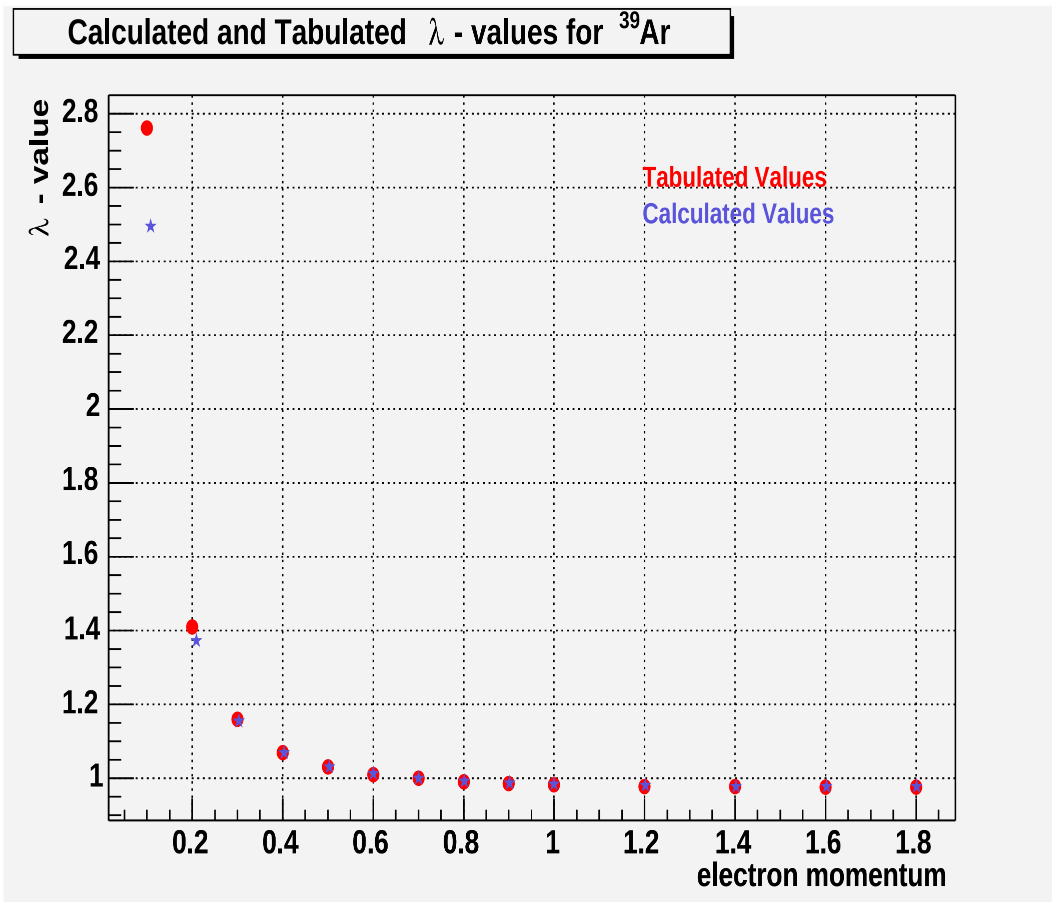

Figure 7: ^{39}Ar λ values.

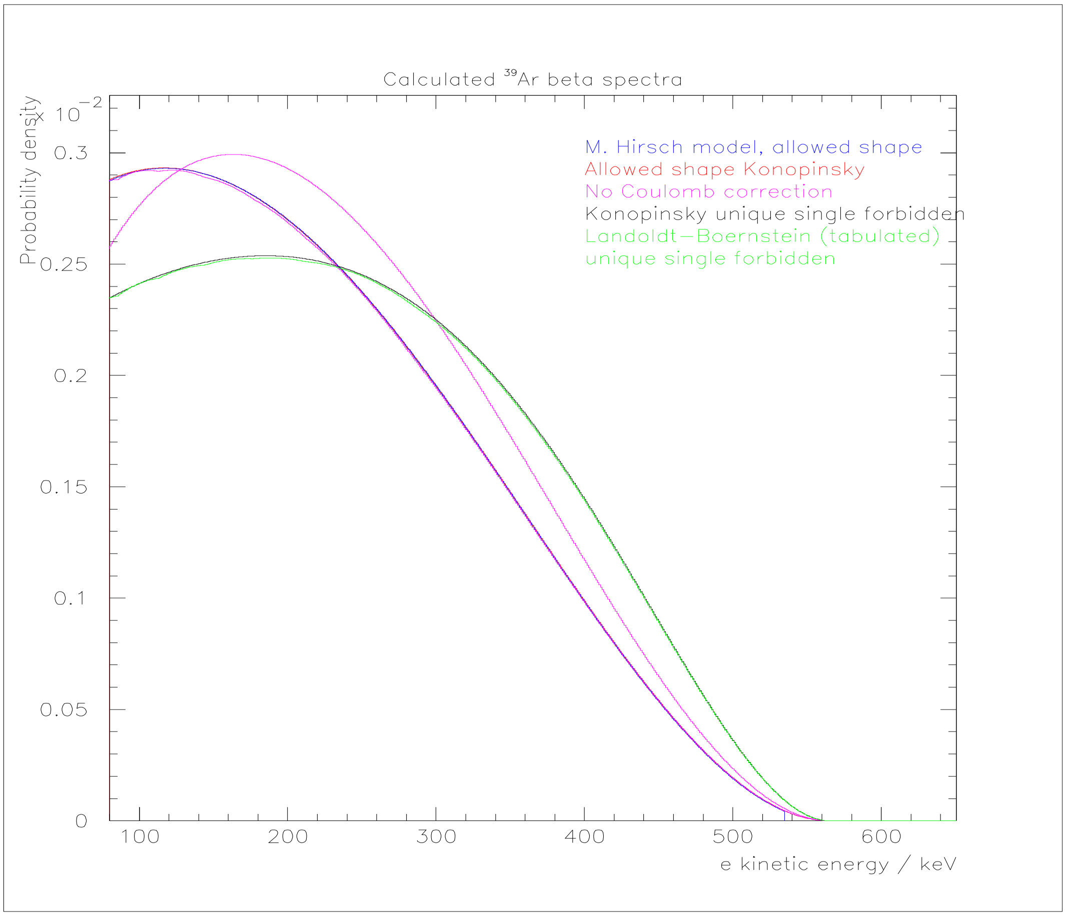

Figure 8: Calculated and tabulated beta spectra for ^{39}Ar.

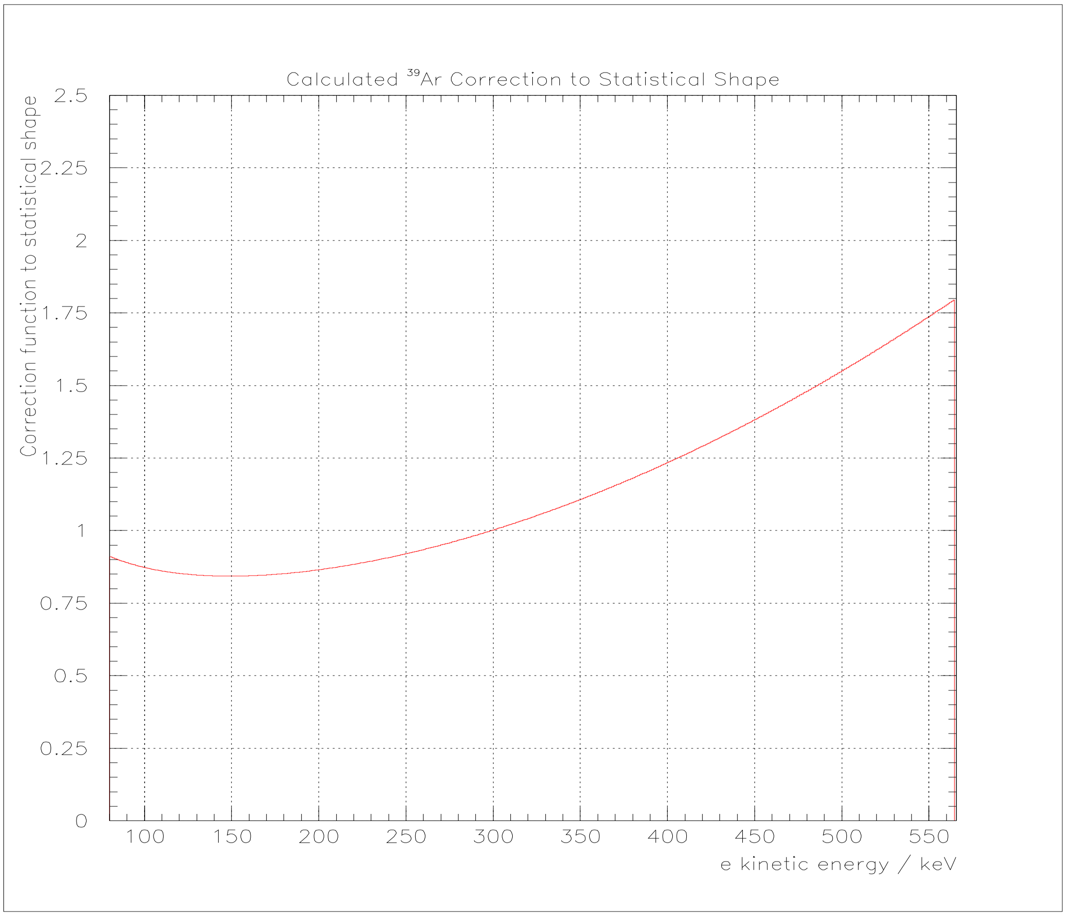

Figure 9: ^{39}Ar correction to statistical shape.

4 The First Forbidden Decay of ^{210}Bi

The beta spectrum for ^{210}Bi is known to deviate from the allowed shape. There have been studies to determine the correction factor and we have taken those provided by Daniel [7]. The form for this correction factor is

$$C(W) = 1 + aW + \frac{b}{W} = cW^2 \tag{8}$$

where a = 0.578, b = 28.466, and c = -0.658 were determined experimentally. Using this correction factor in place of $S(\pm Z, W)$ in Eq. (1) gives the measured beta spectrum for ^{210}Bi as seen in Figure 10. The total correction one needs to apply to the statistical shape is seen in Figure 11.

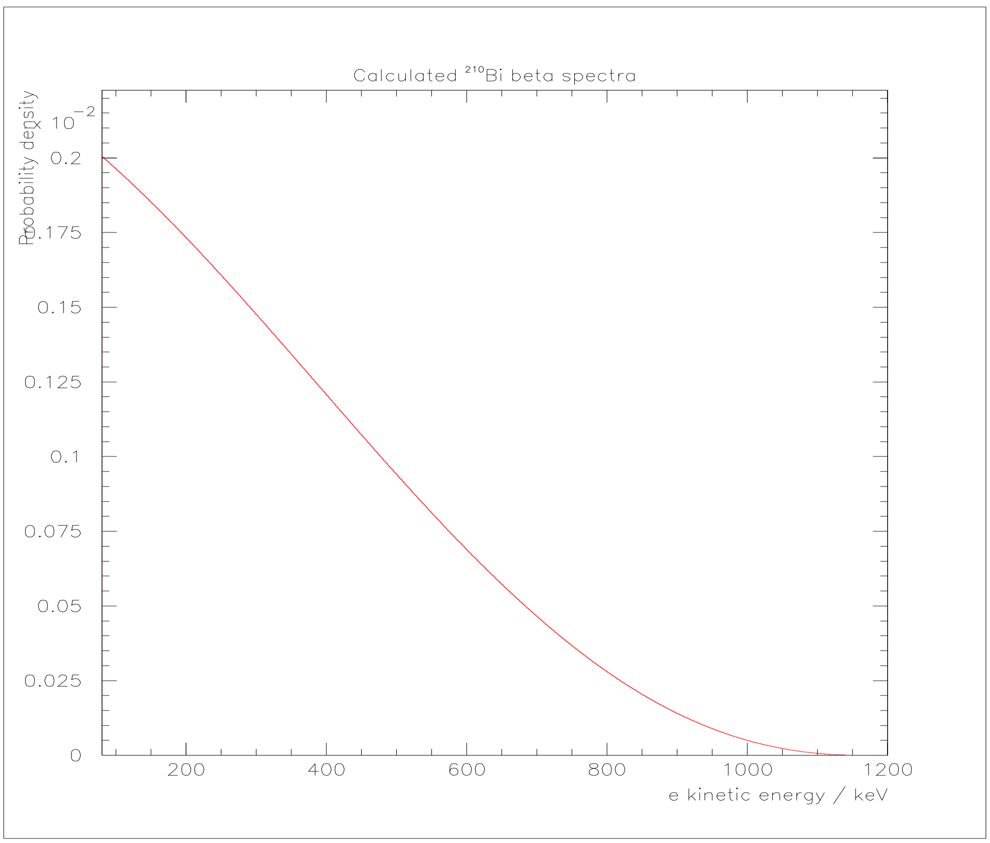

Figure 10: ^{210}Bi beta spectrum.

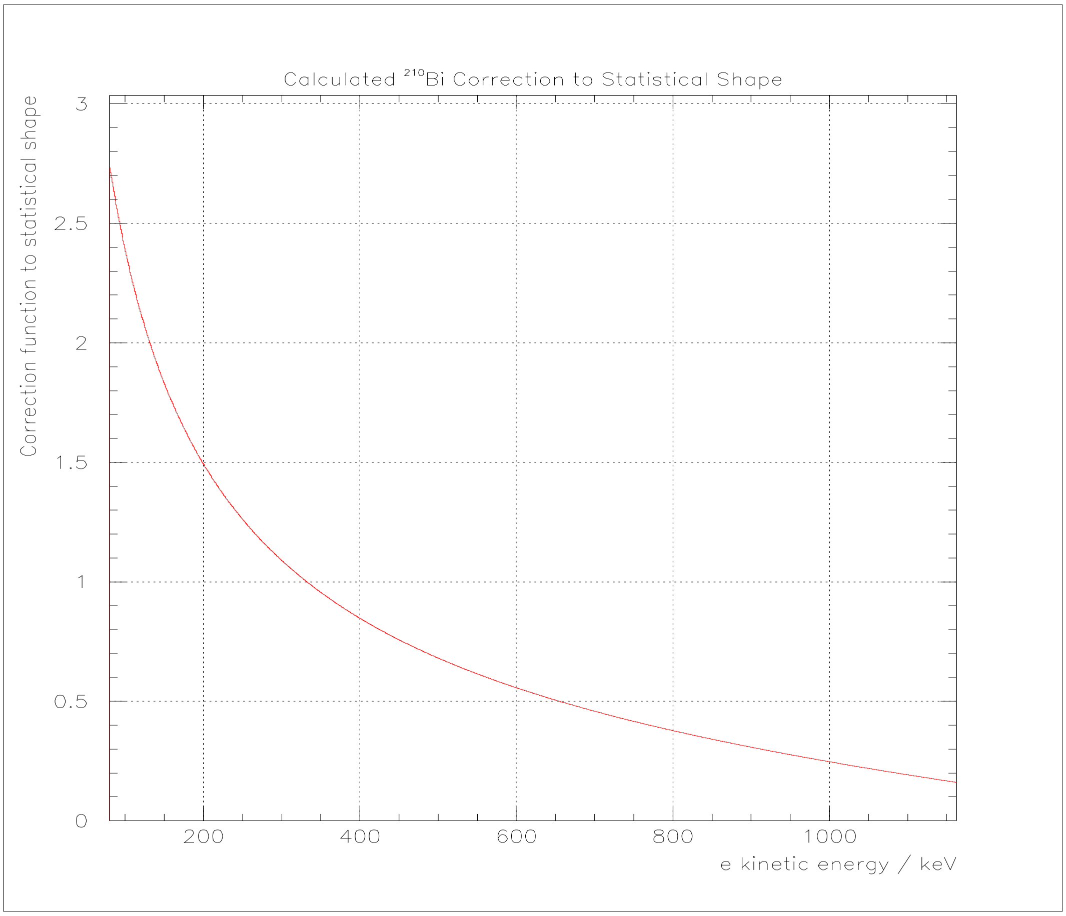

Figure 11: Calculated ^{210}Bi Statistical Shape Correction.

5 Conclusion

Since there is no experimental data available to compare with our calculated forbidden spectra, we were forced to look for agreement with tabulated data. Our spectral corrections agree well with those given in [5]. For the spectrum obtained for ^{210}Bi we took tabulated values obtained directly from experiment to use for the correction factor. All studied decays show significant deviations from the simple statistical shape.

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