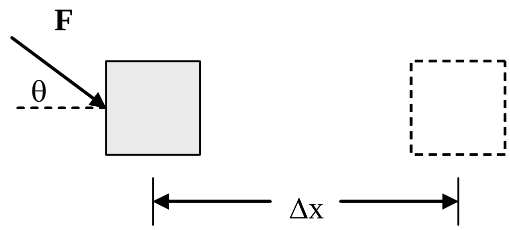
Chapters 7&8 – Work and Energy

Chapters 7 and 8 are about work and energy. Energy is the ability to do work because of an object's state of rest or motion. So let's first consider the concept of work.

Work

For a constant force, work is defined as the product of the force in the direction of the displacement times the displacement. If the force is parallel to the displacement then

$W = F\Delta x$ [N·m = joule (J)]



If the force makes an angle with the displacement then

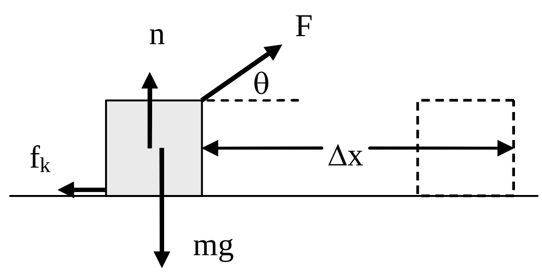
$W = F\Delta x \cos\theta$,

where θ is the angle between the force and the displacement. The above equation can also be expressed in terms of the scalar, or dot, product:

$W = \mathbf{F} \bullet \Delta\mathbf{r}$

If **F** is perpendicular to the displacement ($\theta = 90^o$), then W = 0. If **F** has a component opposite to the displacement ($\theta > 90^o$), then the W is negative. Friction always does negative work on a moving object since the frictional force is opposite to the displacement.

Example:
A 20-kg mass is pulled along a level surface a distance of 4 m by a 150-N force directed 30^o above the horizontal direction. The frictional force acting on the object is 50 N. Find the work done by each individual force acting on the mass and find the total work.



Applied force: $W_F = F\Delta x\cos\theta = (150\text{ N})(4\text{ m})\cos(30^o) = \underline{519.6\text{ J}}$

Normal force: $W_n = n\Delta x\cos(90^o) = \underline{0}$

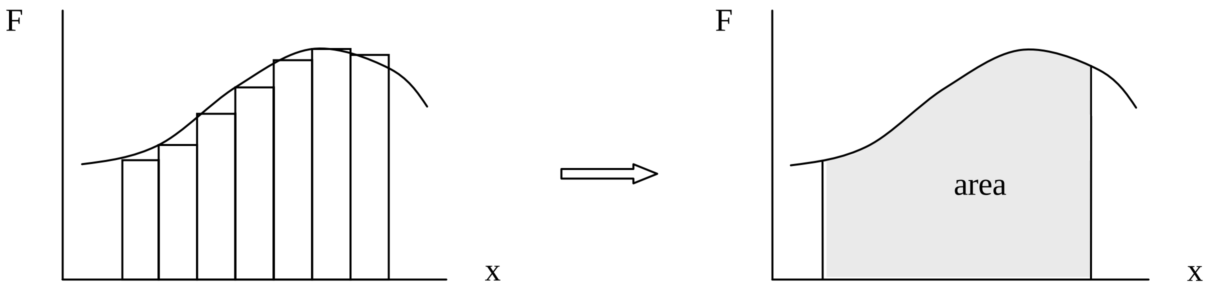
Gravity: $W_g = mg\Delta x\cos(90^o) = \underline{0}$

Friction: $W_f = f_k\Delta x \cos(180^o) = -f_k\Delta x = -(50\text{ N})(4\text{ m}) = \underline{-200\text{ J}}$

Total work: $W = W_F + W_n + W_g + W_n = 519.6\text{ J} - 200\text{ J} = \underline{319.6\text{ J}}$

For a force which varies with position, the net work can be approximated by dividing the displacement into small steps during which the force is nearly constant -

$$W \approx \Sigma F \Delta x.$$



In the limit of infinitesimally small steps,

$$W = \int F dx.$$

Thus, the work is the area under the force versus displacement curve.

Work-Energy Theorem

The net work done on an object is the change in the kinetic energy:

$$W_{net} = \Delta K$$

Kinetic energy is defined as $KE = ½ mv^2$. Thus,

$$W_{net} = ½ mv_f^2 - ½ mv_i^2.$$

The work-energy theorem follows from Newton's 2nd law of motion. For a constant force in the direction of the displacement,

$$W_{net} = F\Delta x = ma\Delta x.$$

Since the acceleration is constant, then $v^2 = v_0^2 + 2a\Delta x$, or $a\Delta x = ½ (v^2 - v_0^2)$. So

$$W_{net} = m ½ (v^2 - v_0^2) = ½ mv^2 - ½ mv_0^2 = \Delta K$$

The same more general result is obtained if the force is not constant:

$$W_{net} = \int F dx = \int madx = \int m(dv/dt)\, dx = \int mvdv = ½\, mv^2 – ½\, mv_0^2.$$

Example: In the above example, assume that the block starts with an initial speed of 5 m/s. What is its speed after it has moved 4 m?

$$½\, mv_f^2 = W_{net} + ½\, mv_i^2 = 319.6 \text{ J} + ½\, (20 \text{ kg})(5 \text{ m/s})^2 = 319.6 \text{ J} + 250 \text{ J} = 569.6 \text{ J}$$

$½\,(20 \text{ kg})v_f^2 = 569.6$ J, $\underline{v_f = 7.5 \text{ m/s}}$

Potential Energy

The potential energy of a mechanical system is related to its position. That is, the position of an object in a system may allow it to do work.

Gravitational potential energy

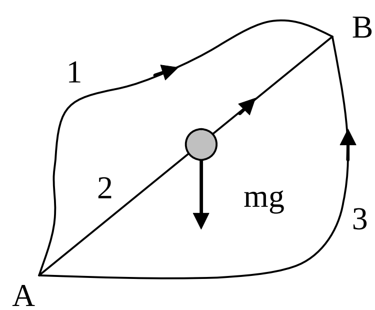
An object under that influence of a gravitational force will have potential energy relative to some fixed position. If an object falls a distance h, then the work done by gravity is

$$W_g = Fh = mgh$$

This work done by gravity on a falling object is the amount by which the gravitational potential energy is reduced. The work that you would do in lifting the object against the force of gravity would be the increase in potential energy, which is abbreviated as PE = U. The potential energy is always defined relative to some fixed position. So if this fixed position is y = 0, then

$$U = mgy$$

Conservative and Nonconservative Forces



A potential energy can only be associated with a force if it is *conservative*. A conservative force is one for which the work done by the force on an object as it goes from point A to point B doesn't depend on the path taken. The work done by gravity, for example, doesn't depend on path since any horizontal motion is perpendicular to the force and doesn't contribute to the work.

In the illustration to the right, the work done by gravity as the object moves from A to B is the same for paths 1, 2 and 3. Another way to describe a conservative force is to say

that the net work done by the force if the object goes in a closed path (e.g., from A to B and back to A) is zero.
The change in potential energy associated with a conservative force is just the negative of the work done by the force. That is,

$$\Delta U = -W = -\int_1^2 \mathbf{F} \cdot d\mathbf{r}$$

A *nonconservative* force is one for which the work depends on the path. Friction is a nonconservative force. For example, if you slide a block on a surface along a closed path the work is not zero.

Conservation of Mechanical Energy

The total energy of a system is the sum of the kinetic and the potential.

$$E = K + U$$

In the absence of friction, the total energy of a *closed* system does not change with time. That is,

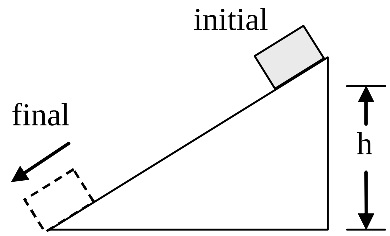
$$\Delta E = \Delta K + \Delta U = \Delta K - W = 0$$

This is the principle of *conservation of energy*.

(If the system is not closed, then energy may be transferred into or out of the system and the total energy of the system may not be constant.)

Example:

A block slides from rest down a frictionless inclined plane of height h. What is its speed at the base of the incline?



$$E_{initial} = E_{final}$$

$$K_i + U_i = K_f + U_f$$

$$\tfrac{1}{2} mv_i^2 + mgy_i = \tfrac{1}{2} mv_f^2 + mgy_f$$

Choose the base of the incline as y = 0. (The choice doesn't really matter as long as we are consistent.) Then $y_i = h$ and $y_f = 0$ and

$$0 + mgh = \tfrac{1}{2} mv_f^2$$

and $v_f = \sqrt{2gh}$

This speed is the same as would be obtained if the mass were dropped straight down from a height h. All that matters is the change in height. (Of course the time to slide down would be longer than the time to fall.)

Example:

A projectile is fired into the air with speed v_0. What is its speed when it reaches a height h? Using conservation of energy,

$$E_f = E_i$$

$$\tfrac{1}{2} mv_f^2 + mgy_f = \tfrac{1}{2} mv_i^2 + mgy_i$$

Cancelling out m, we can rewrite this as

$$v^2 = v_0^2 – 2g(y_f-y_i) = v_0^2 – 2gh$$

or, $$v = \sqrt{v_0^2 - 2gh}$$

This is the same result we would have gotten using the projectile equations that were developed in Chapter 3:

$$v_x = v_{x0},\ v_y^2 = v_{y0}^2 -2g\Delta y$$

$$v_x^2 + v_y^2 = v_{x0}^2 + v_{y0}^2 -2g\Delta y$$

or, $$v^2 = v_0^2 – 2g\Delta y$$

Potential Energy Stored in a Spring

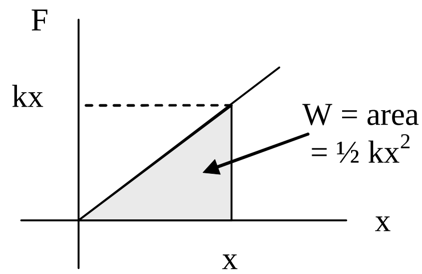
Many springs obey Hooke’s law,

$$F = -kx$$

where F is the restoring force of the spring, x is the stretch (or compression) of the spring from its equilibrium position, and k is the force constant of the spring (a measure of the spring stiffness). The minus sign signifies that the restoring force is opposite to the displacement.

The spring force is a conservative force and a potential energy can be associated with the stretch or compression of the spring.

The graph to the right shows the force required to stretch a spring (opposite to the restoring force) as a function of the stretch. The potential energy stored in the stretched spring is the work required to stretch it. For a constant force W = Fx. If the force is not constant, then we use the average force. For the spring,



$$F_{ave} = ½ (F_i + F_f) = ½ (k(0) + kx) = ½ kx, \text{ and } W = F_{ave}x = ½ kx^2$$

(Note: Using the average force is the same as taking the work to be the area under the F versus x curve.)

Thus, $U = ½ kx^2$

Example:

A spring gun has a force constant k = 200 N/m. It is used to fire a 10-g projectile horizontally. If the spring is compressed 7 cm in the cocked position, what is the speed of the projectile when it leaves the barrel of the gun?

We solve this using conservation of energy. We don't need to consider gravitational potential energy since the height of the projectile is the same in the cocked position and when it exits the barrel.

$$E_f = E_i$$

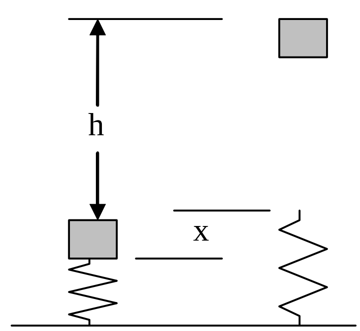
$$½ mv_f^2 + ½ kx_f^2 = ½ mv_i^2 + ½ kx_i^2$$

$$½ mv^2 + ½ k(0)^2 = ½ m(0)^2 + ½ kx^2$$

$$v = x\sqrt{\frac{k}{m}} = 0.07\,m\sqrt{\frac{200\,N/m}{0.01\,kg}} = 9.9\,m/s$$

Example:

This same spring gun is fired straight up into the air. How high does it go?



$$E_f = E_i$$

$$½ mv_f^2 + mgy_f + ½ kx_f^2$$

$$= ½ mv_i^2 + mgy_i + ½ kx_i^2$$

$$0 + mgh + 0 = 0 + 0 + ½ kx^2$$

$$h = \frac{\frac{1}{2}kx^2}{mg} = \frac{\frac{1}{2}(200)(0.07)^2}{(0.01)(9.8)} = 5.0\,m$$

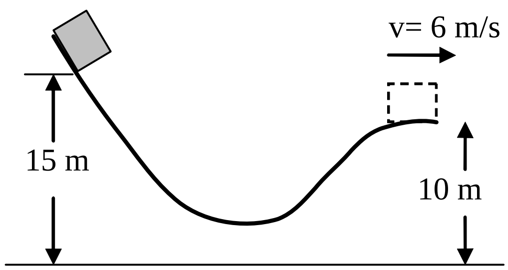
Frictional Forces and Energy

If friction or some other nonconservative force is present, then energy is not conserved. In this case,

$$W_f = \Delta E = E_f - E_i$$

Since W_f is negative, this means that friction will reduce the total mechanical energy. (Some of the mechanical energy is converted into internal thermal energy.)
Example:

A 2-kg block is released from rest on a slide 15 m above the ground. It leaves the end of the slide 10 m above the ground with a speed of 6 m/s. How much work was done by friction on the block?



$$W_f = (KE_f + PE_f) - (KE_i + PE_i) = (\tfrac{1}{2} mv_f^2 + mgy_f) - (\tfrac{1}{2} mv_i^2 + mgy_i)$$

$$= [\tfrac{1}{2}(2)(6)^2 + (2)(9.8)(10)] - [\tfrac{1}{2}(2)(0)^2 + (2)(9.8)(15)]$$

$$= 36 + 196 - 294 = \underline{-62\text{ J}}$$

Power

Power is the rate at which work is done.

$$P = \frac{W}{\Delta t}$$ Units are J/s = watt (W)

Since $W = F\Delta x$, then $P = \frac{F\Delta x}{\Delta t} = F\bar{v}$

Example:

What is the average power required to lift a 60-kg person a height of 2 m in 5 seconds?

$$P = \frac{W}{\Delta t} = \frac{F\Delta y}{\Delta t} = \frac{mg\Delta y}{\Delta t} = \frac{(60)(9.8)(2)}{5} = 235.2\ W$$

Example:

The power company bills its customers based on the number of *kilowatts* of energy used. Suppose the energy cost is 10 cents per kwh. How much would it cost to keep a 75 W bulb lamp on for one month?

Energy = E = PΔt

Assuming 30 days in the month, Δt = (30 d)(24 hr/d) = 720 hr. Power = P = 0.075 kW.

Then E = (0.075 kW)(720 hr) = 54 kwh, and

Cost = ($0.10/kwh)(54 kwh) = <u>$5.40</u>

So, if you had 10 such bulbs on all the time, it would add up to ??? in a year?

Relationship between conservative forces and potential energy

Since the change in potential energy is the negative of the work done by a conservative force, then this force can be determined from the potential energy. In 1-D,

$$\Delta U = U(x) - U(x_0) = -\int_{x_0}^{x} F_x dx$$

This means that

$$F_x = -\frac{dU}{dx}$$

For example, the gravitation potential energy is U = mgy. Thus, the force of gravity is

$$F_y = -\frac{dU}{dy} = -\frac{d(mgy)}{dy} = -mg$$

And for a spring, $U = \frac{1}{2}kx^2$, so $F = -\frac{d(\frac{1}{2}kx^2)}{dx} = -kx$.