

## Chapter 9 – Linear Momentum and Collisions

This chapter is about the concept of *linear momentum*.

**Linear momentum** is defined as

$$\mathbf{p} = m\mathbf{v} \quad [\text{kg}\cdot\text{m/s}]$$

Momentum is a *vector* whose direction is the same as the velocity. So, in 2-D

$$p_x = mv_x, \quad p_y = mv_y$$

Momentum is *not* the same as kinetic energy.  $K = \frac{1}{2}mv^2$  is a *scalar* and has no direction. Also, K depends on the square of the speed. Because of this difference, two different objects can have the same momentum but have different kinetic energies.

*Example:*

Car A has a mass of 2000 kg and is traveling at 10 m/s north. Car B has a mass of 1000 kg and is traveling at 20 m/s north. Both cars have the same momentum –

$$p_A = (2000 \text{ kg})(10 \text{ m/s}) = 20,000 \text{ kg}\cdot\text{m/s}, \text{ direction} = \text{north}$$

$$p_B = (1000 \text{ kg})(20 \text{ m/s}) = 20,000 \text{ kg}\cdot\text{m/s}, \text{ direction} = \text{north}$$

However, they have different kinetic energies -

$$K_A = \frac{1}{2} (2000 \text{ kg})(10 \text{ m/s})^2 = 100,000 \text{ J}$$

$$K_B = \frac{1}{2} (1000 \text{ kg})(20 \text{ m/s})^2 = 200,000 \text{ J}$$

Newton's 2<sup>nd</sup> law can be written in terms of change in momentum:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

This is the same as  $\mathbf{F} = m\mathbf{a}$ , since

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m \frac{d\mathbf{v}}{dt} = m\mathbf{a} .$$

### Impulse

Impulse is defined as

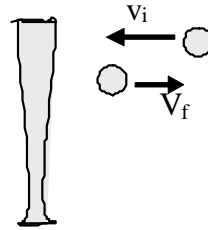
$$\text{Impulse} = \int \mathbf{F}dt = \mathbf{F}_{\text{ave}}\Delta t \quad [\text{N}\cdot\text{s} = \text{kg}\cdot\text{m/s}]$$

The impulse exerted by a force depends on the time of application and the average force during that time. From Newton's 2<sup>nd</sup> law, the impulse on an object is the same as its change in momentum:

$$\mathbf{F}_{\text{ave}}\Delta t = \Delta\mathbf{p} = m\mathbf{v}_f - m\mathbf{v}_i$$

*Example:*

A pitcher throws a baseball toward a batter with a speed of 40 m/s toward a batter. The batter hits the ball straight back towards the pitcher with a speed of 30 m/s. A baseball has a mass of about 145 g.



What is the impulse exerted on the ball by the bat?

Assume right = + , left = -

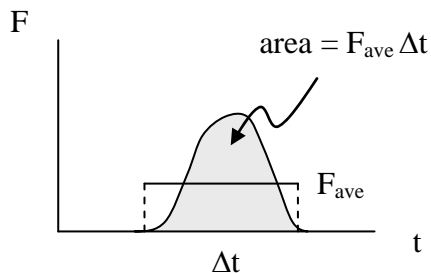
$$\begin{aligned} \text{Impulse} &= \Delta p = mv_f - mv_i = m(v_f - v_i) = (0.145 \text{ kg})(30 \text{ m/s} - (-40 \text{ m/s})) \\ &= (0.145 \text{ kg})(70 \text{ m/s}) = \underline{10.15 \text{ kg}\cdot\text{m/s}} \end{aligned}$$

If the contact time was 5 milliseconds, what was the average force exerted by the bat on the ball?

$$F = \frac{\Delta p}{\Delta t} = \frac{10.15 \text{ kg} \cdot \text{m/s}}{0.005 \text{ s}} = \underline{2030 \text{ N}}$$

### Impulse from a variable force

If the force varies during a contact, then the impulse depends on the average force during the contact. This is equivalent to finding the impulse from the area under the *force versus time* graph.



(A point of possible confusion - The above is similar to, but not the same as, the work energy theorem, which states that  $\mathbf{F}_{\text{ave}}\Delta x$  (= area under F vs x curve) =  $\Delta\text{KE}$ .)

## Reducing the impact force during a collision

The damage that results from a collision is a consequence of the size of the impact force. So, to reduce the damage we need to spread out the time of impact.

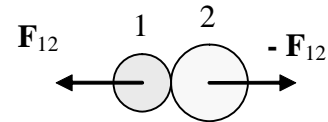
*Example:*

An egg dropped from a height of 1 m onto a concrete floor will break, but it will not break if dropped from the same height onto a pillow. Why? The impulse is the same in both cases (same  $\Delta\mathbf{p}$ ). The pillow increases time of contact ( $\Delta t$ ) and thus reduces the average force ( $\mathbf{F}$ ).

Other examples would include a safety airbag in a car, a padded dashboard, a flexible bumper on a car, ...

## Conservation of Linear Momentum

Consider a collision between two balls. During the collision the balls exert equal and oppositely directed forces on each other (Newton's 3<sup>rd</sup> law). That is,



$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

Since the contact times are the same for both masses,

$$\mathbf{F}_{12} \Delta t = -\mathbf{F}_{21} \Delta t$$

If all other forces can be neglected, then from Newton's 2<sup>nd</sup> law

$$\Delta\mathbf{p}_1 = -\Delta\mathbf{p}_2$$

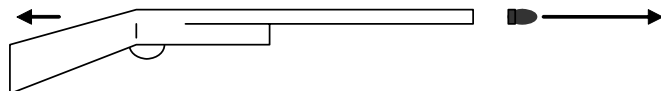
$$\Delta(\mathbf{p}_1 + \mathbf{p}_2) = \Delta\mathbf{p}_{\text{total}} = 0$$

Or,  $\mathbf{p}_{\text{total}} = \text{constant}$

The above argument applies to an arbitrary number of particles. If  $\mathbf{F}_{\text{ext}} = 0$ , then the total momentum of a system remains constant in time.

*Example:*

A 10-kg gun fires a 25-g bullet with a speed of 300 m/s. What would be the recoil speed of the gun? Assume the gun is held lightly so that its recoil is not restricted.



$$p_f = p_i$$

$$m_b v_b + m_g v_g = 0$$

$$v_g = -m_b v_b / m_g = -(0.025)(300) / 10 = \underline{-0.75 \text{ m/s}}$$

Note: If you hold the gun tightly, its recoil speed will be much less since your mass adds to the gun in the above calculation.

## Collisions

Collisions can be classified as elastic or inelastic. In an **elastic collision** kinetic energy is conserved. In an **inelastic collision** kinetic energy is *not* conserved. When two colliding objects stick together the collision is referred to as **completely inelastic**. In a completely inelastic collision you have the maximum loss of kinetic energy; however, not all the kinetic energy is necessarily lost.

Two colliding billiard balls may be nearly elastic (but not completely). If you throw a piece of putty and it sticks to a wall, then the collision is completely inelastic.

Note: Whether the collision is elastic or inelastic, momentum is *always* conserved.

*Example:*

A 5000-kg truck traveling at 10 m/s makes a head-on collision with a 1000-kg car traveling at 30 m/s. The car and truck become entangled and stick together on impact. What is their common velocity immediately after the collision?

Choose the initial direction of motion of the truck as the positive direction. Then

$$p_f = p_i$$

$$(m_t + m_c)v_f = m_t v_t + m_c v_c$$

$$v_f = \frac{m_t v_t + m_c v_c}{m_t + m_c} = \frac{(5000)(10) + (1000)(-30)}{5000 + 1000} = \underline{3.33 \text{ m/s}}$$

Note that we used -30 m/s for the velocity of the car. The fact that we got +3.3 m/s means that the entangled vehicles move in the direction of the truck's initial velocity.

How much kinetic energy was lost in the above collision?

$$K_i = \frac{1}{2}m_t v_t^2 + \frac{1}{2}m_c v_c^2 = \frac{1}{2}(5000)(10)^2 + \frac{1}{2}(1000)(30)^2 = 2.5 \times 10^5 J + 4.5 \times 10^5 J$$

$$= 7 \times 10^5 J$$

$$K_f = \frac{1}{2}(m_t + m_c)v_f^2 = \frac{1}{2}(5000 + 1000)(3.33)^2 = 3.33 \times 10^4 J$$

$$\Delta K = K_f - K_i = 3.33 \times 10^4 - 7 \times 10^5 = -6.67 \times 10^5 J$$

So,  $K(\text{lost}) = 6.67 \times 10^5 J$ .

*Example:*

A 1-kg ball traveling at 8 m/s to the right makes a head-on collision with a 3-kg ball traveling at 2 m/s to the left. After the collision the 3-kg ball moves to the right with speed 2.5 m/s. What is the speed and direction of motion of the 1-kg ball after the collision?

$$P_f = P_i$$

$$m_1 v_{1f} + m_2 v_{2f} = m_1 v_{1i} + m_2 v_{2i}$$

$$v_{1f} = \frac{m_1 v_{1i} + m_2 v_{2i} - m_2 v_{2f}}{m_1} = \frac{(1\text{kg})(8\text{m/s}) + (3\text{kg})(-2\text{m/s}) - (3\text{kg})(2.5\text{m/s})}{1\text{kg}}$$

$$= \underline{\underline{-5.5\text{m/s}}}$$

So, the 1-kg ball rebounds to the left with speed 5.5 m/s.

Was this an elastic collision?

$$K_i = \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}(1)(8)^2 + \frac{1}{2}(3)(2)^2 = 38 J$$

$$K_f = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 = \frac{1}{2}(1)(5.5)^2 + \frac{1}{2}(3)(2.5)^2 = 24.5 J$$

Kinetic energy was *not* conserved, so this was an *inelastic* collision, even though the two balls bounced from each other.

Suggested practice exercise: Show that if the 3-kg ball rebounds with a speed of 3 m/s, then the collision is elastic.

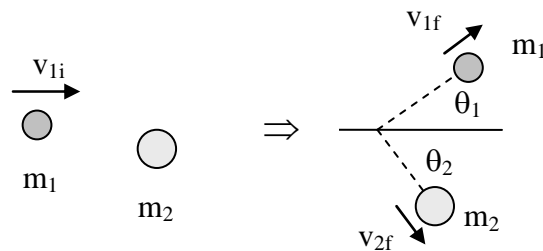
## 2-D collisions

Since momentum is a vector, it must be conserved in all directions. For a 2-D collision

$$p_{xf} = p_{xi} \Rightarrow m_1 v_{1xf} + m_2 v_{2xf} = m_1 v_{1xi} + m_2 v_{2xi}$$

$$p_{yf} = p_{yi} \Rightarrow m_1 v_{1yf} + m_2 v_{2yf} = m_1 v_{1yi} + m_2 v_{2yi}$$

*Example:* A 2-kg ball traveling to the right at 10 m/s makes a glancing collision with a 3-kg ball initially at rest. After the collision the 2-kg ball travels with speed 8.4 m/s at an angle  $37^\circ$  above the initial direction. What is the speed and direction of the 3-kg ball after the collision?



$$p_{xf} = p_{xi}$$

$$m_1 v_{1f} \cos \theta_1 + m_2 v_{2f} \cos \theta_2 = m_1 v_{1i}$$

$$(2\text{kg})(8.4\text{m/s}) \cos(37^\circ) + (3\text{kg})v_{2f} \cos \theta_2 = (2\text{kg})(10\text{m/s})$$

$$\underline{v_{2f} \cos \theta_2 = 2.19\text{m/s}} \quad (\text{Eq. 1})$$

$$p_{yf} = p_{yi}$$

$$m_1 v_{1f} \sin \theta_1 + m_2 v_{2f} \sin \theta_2 = 0$$

$$(2\text{kg})(8.4\text{m/s}) \sin(37^\circ) + (3\text{kg})v_{2f} \sin \theta_2 = 0$$

$$(\text{Eq. 2})$$

$$\underline{v_{2f} \sin \theta_2 = -3.37\text{m/s}} \quad (\text{Eq. 2})$$

In Eqs. 1 and 2 above, we have 2 unknowns,  $v_{2f}$  and  $\theta_2$ . We can solve as follows:

From Eq. 1,  $v_{2f} = \frac{2.19}{\cos \theta_2}$ . Substituting this into Eq. 1,

$$\left( \frac{2.19}{\cos \theta_2} \right) \sin \theta_2 = -3.37$$

$$\tan \theta_2 = \frac{-3.37}{2.19}, \quad \theta_2 = -57^\circ$$

Then from Eq. 1,

$$v_{2f} = \frac{2.19}{\cos(57^\circ)} = \underline{4.02 \text{ m/s}}$$

Note: If you calculate the initial kinetic energy and final kinetic energy, you will see that this collision was not elastic.

*Example:* Consider a collision like that in the example above, but assume that it is *elastic*. Then,

$$m_1 \mathbf{v}_{1i} = m_1 \mathbf{v}_{1f} + m_2 \mathbf{v}_{2f} \quad \dots(1)$$

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad \dots(2)$$

If the balls have the same mass, as in billiards, then we have

$$\mathbf{v}_{1i} = \mathbf{v}_{1f} + \mathbf{v}_{2f} \quad \dots(3)$$

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2 \quad \dots(4)$$

From the first of the above equations, we can write

$$\mathbf{v}_{1i} \cdot \mathbf{v}_{1i} = (\mathbf{v}_{1f} + \mathbf{v}_{2f}) \cdot (\mathbf{v}_{1f} + \mathbf{v}_{2f}), \text{ or}$$

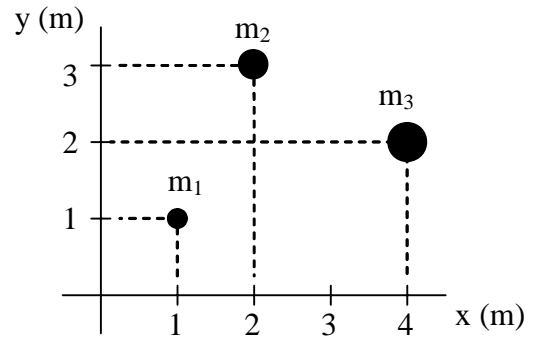
$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2 + 2\mathbf{v}_{1f} \cdot \mathbf{v}_{2f} \quad \dots(5)$$

Comparing equations (4) and (5), we see that  $\mathbf{v}_{1f} \cdot \mathbf{v}_{2f} = 0$ . This means that the final velocities are perpendicular ( $\theta_1 + \theta_2 = 90^\circ$ ).

## Center of Mass

The center of mass of a system of objects is the mass weighted average position of the objects. In a uniform gravitational field this 'center of mass' is the same as the 'center of gravity'. In 2-D the location of the center of mass is given by

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i}, \quad y_{cm} = \frac{\sum m_i y_i}{\sum m_i}$$



### Example:

What is the location of the center of mass of the three particles in the figure to the right?

Given:  $m_1 = 10 \text{ g}$ ,  $m_2 = 20 \text{ g}$ ,  $m_3 = 30 \text{ g}$ .

$$\begin{aligned} x_{cm} &= \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \\ &= \frac{(0.01\text{kg})(1\text{m}) + (0.02\text{kg})(2\text{m}) + (0.03\text{kg})(4\text{m})}{0.01\text{kg} + 0.02\text{kg} + 0.03\text{kg}} = \underline{2.83\text{m}} \\ y_{cm} &= \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{(0.01\text{kg})(1\text{m}) + (0.02\text{kg})(3\text{m}) + (0.03\text{kg})(2\text{m})}{0.01\text{kg} + 0.02\text{kg} + 0.03\text{kg}} \\ &= \underline{2.17\text{m}} \end{aligned}$$

If a regularly shaped object has a uniform mass density, then its center of mass is the same as its geometrical center.