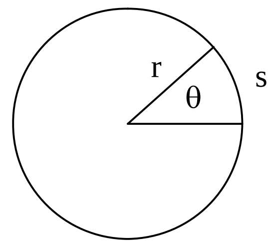
Chapter 10 & 11 – Rotational Motion

Chapters 10 and 11 involve rotational kinematics and dynamics. Rotational kinematics relates rotational position, velocity, acceleration and time. Rotational dynamics deals with rotational energy, angular momentum, and torque.

Angular displacement, velocity, and acceleration



Angle (θ) can be defined in terms of radius (r) and arc length (s) on a circle as

$$\theta = \frac{s}{r}$$

Unless otherwise stated, θ is usually measured in the counterclockwise direction from the positive x-axis. If *s* and *r* are measured in the same units (e.g., m), then the above equation gives θ is in *radians*. Angle can also be measured in degrees or revolutions.

$$1 \text{ rev} = 2\pi \text{ rad} = 360^{o}$$

Average *angular velocity* is given by

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

and instantaneous angular velocity is

$$\omega = \frac{d\theta}{dt}$$

The instantaneous angular velocity is just the average angular velocity in the limit of very short time interval. Depending on units for θ and t, units for ω can be rad/s, deg/s, rev/s, rev/min (rpm), etc.

Average *angular acceleration* is given by

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

and instantaneous angular acceleration is

$$\alpha = \frac{d\omega}{dt}$$

Units for angular acceleration can be rad/s^2, deg/s^2, rev/s^2, rev/min^2, etc.

The relationships above are mathematically similar to those for motion in 1-D. Thus, we can get the equations for constant angular acceleration from those for constant linear acceleration by appropriately changing the variables.

1-D motion with constant a		Rot. motion with constant α
$\Delta x = v_0 t + \frac{1}{2} a t^2$	$x \to \theta$	$\Delta \theta = \omega_0 t + \frac{1}{2} \alpha t^2$
$v = v_0 + a t$	$v \to \omega$	$\omega = \omega_0 + \alpha t$
$v^2 = v_0^2 + 2 a \Delta x$	$a \to \alpha$	$\omega^2 = \omega_0^2 + 2 \alpha \Delta \theta$

It is important that the units in the above equations be compatible. For example, if α is in rad/s^2, then ω should be in rad/s, t in s, and you will get θ in rad. Also, *don't* mix the left and right equations above. For example, the equation $\Delta \theta = v_0 t + \frac{1}{2} \alpha t^2$ would not make sense. Also, don't confuse α and *a*, although they look similar.

Example:

A wheel increases its rotational velocity from 200 rpm to 300 rpm in 10 sec.

What is its angular acceleration in rad/s^2?

$$\omega_0 = (200\, rev/\min)(1\min/60s)(2\pi\, rad/rev) = 20.9\, rad/s$$

$$\omega = (300\, rev/\min)(1\min/60s)(2\pi\, rad/rev) = 31.4\, rad/s$$

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{\Delta t} = \frac{31.4\, rad/s - 20.9\, rad/s}{10\, s} = \underline{1.05\, rad/s^2}$$

How many turns did the wheel make?

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = (20.9\, rad/s)(10\, s) + \frac{1}{2}(1.05\, rad/s^2)(10\, s)^2 = 261.5\, rad$$

$$no.turns = \frac{\theta}{2\pi} = \underline{41.6}$$

$$(\ Or,\ use\ \theta = \overline{\omega} t = \frac{(\omega_0 + \omega)}{2} t\ )$$

Relationships between angular and linear motion

Since $s = r\theta$, then it follows that

$$\frac{ds}{dt} = r\frac{d\theta}{dt}, \quad \text{or} \quad v_t = r\omega$$

$$\frac{dv_t}{dt} = r\frac{d\omega}{dt}, \quad \text{or} \quad a_t = r\alpha$$

In the above, v_t is the tangential speed of a point going around a circle and a_t is the component of acceleration tangent to the circle.

<u>Example</u>:

A merry-go-round rotates at a constant angular speed. It takes 20 sec to make a complete revolution. What is the speed of a rider who is 4 m from the center?

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\, rad}{20\, s} = 0.314\, rad/s$$

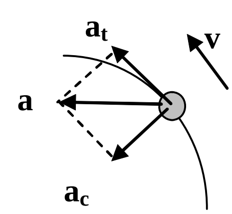
$$v_t = r\omega = (4\,m)(0.314\, rad/s) = \underline{1.26\, m/s}$$

The speed of a rider increases as he moves further from the center.

Combined tangential and centripetal acceleration

We have previously shown that an object moving in a curved path can have both a tangential acceleration, as discussed above, and a radial (or centripetal) acceleration.

$$a_t = \frac{dv}{dt} \qquad\qquad a_r = \frac{v^2}{r}$$



The total acceleration is the vector sum:

$$a = a_r + a_t$$

Since these two components of the acceleration are mutually perpendicular, then the magnitude of the total acceleration is

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

Example:

The merry-go-round in the above example slows uniformly and comes to rest in a time of 2 minutes.

What the total acceleration of a rider when his tangential speed is 1 m/s?

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{(0 - 0.314)\,rad}{60\,s} = -0.0523\,rad/s^2$$

$$a_t = r\alpha = (4m)(-0.0523 rad/s^2) = -0.209\,m/s^2$$

$$a_r = \frac{v^2}{r} = \frac{(1m/s)^2}{4m} = 0.25\,m/s^2$$

$$a = \sqrt{(0.209)^2 + (0.25)^2} = 0.326\,m/s^2$$

The angle that the acceleration makes with the radial direction is

$$\theta = \tan^{-1}\left(\frac{a_t}{a_r}\right) = \tan^{-1}\left(\frac{-0.209}{0.25}\right) = -40^o$$

The angle is in the 'backward' direction.

Rotational Kinetic Energy

For a rigid object rotating about an axis, all the pieces that make up the object rotate in a circle with the same angular velocity ω. Thus, the kinetic energy is

$$K_{rot} = \sum \tfrac{1}{2} m v^2 = \sum \tfrac{1}{2} m\omega^2 r^2 = \tfrac{1}{2}\left(\sum m r^2\right)\omega^2 = \tfrac{1}{2} I\,\omega^2$$

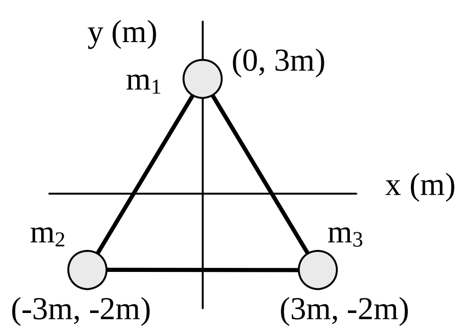
where,

$$I = \sum m r^2,$$

is defined as the *rotational inertia* of the object.

Example:

Three masses, each 2 kg, are located at the corners of a triangle consisting of light rigid rods, as shown to the right. The location (x, y) of each mass is shown. What is the rotational inertia about the x-axis?



What matters here is the nearest distance of each mass from the axis of rotation. So,

$$I_x = \sum mr^2 = m_1 y_1{}^2 + m_2 y_2{}^2 + m_3 y_3{}^2 = (2)(3)^2 + (2)(2)^2 + (2)(2)^2 = \underline{34\,kg \cdot m^2}$$

If the triangle of masses rotates about the x-axis at 60 rpm, what is the rotational kinetic energy?

The angular velocity must be in rad/s, so

$$\omega = (60rpm)(1\text{min}/60s)(2\pi\, rad/rev) = 6.28\, rad/s$$
$$K_{rot} = \tfrac{1}{2} I\omega^2 = \tfrac{1}{2}(34kg \cdot m^2)(6.28\, rad/s)^2 = 670\, J$$

Suggested exercise: Find the rotational inertia and rotational kinetic energy about the y-axis and about the z-axis (which is perpendicular to the page) for the same rotational velocity

Rotational Inertia of a Continuous Mass Distribution

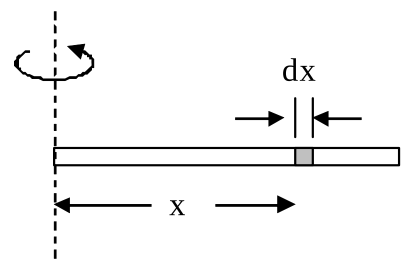
For a continuous distribution of mass,

$$I = \sum \Delta m r^2 \rightarrow \int r^2 dm = \int \rho r^2 dV$$

where ρ is the mass density.

Example:

What is the rotational inertia of a uniform rod about an axis passing through the end and perpendicular to the rod?



$$dm = \frac{M}{L} dx$$

$$I = \int x^2 dm = \int_0^L x^2 \frac{M}{L} dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \tfrac{1}{3} ML^2$$

For a perpendicular axis passing through the center of the rod,

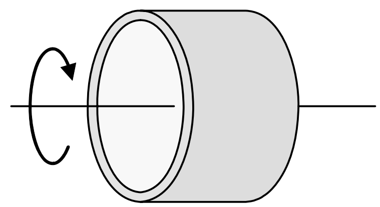
$$I = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx = \tfrac{1}{12} ML^2$$

As the above example shows, the rotational inertia of an object depends about the axis of rotation. For a given orientation, the rotational inertia is smallest if the axis is through the center of mass. For any other axis that is parallel with this axis through the center of mass, the rotational inertia is given by the *parallel axis theorem*,

$$I = I_{cm} + MD^2$$

where D is the distance between the two axes. (You can check to see that this is true for the rod example.)

Rotational Intertia of Various Rolling Objects



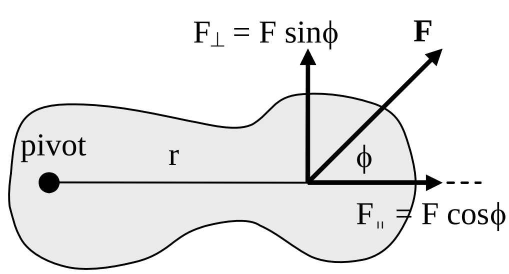
A round object that rolls also rotates about an axis through it center of mass. Below are expressions for some such rotational inertias that have been determined by integrating over the volume of the object.

Object	Rotational Inertia
cylindrical shell	$I = M R^2$
solid cylinder	$I = \frac{1}{2} M R^2$
spherical shell	$I = \frac{2}{3} M R^2$
solid sphere	$I = \frac{2}{5} M R^2$

Torque

Torque can cause an object to rotate. The torque depends on the force and on the point at which the force is applied relative to the axis of rotation. Specifically, torque is the product of the force and the nearest distance between the *line of action* of the force and the *axis or rotation*,

$$\tau = Fd$$ (units are N-m)



If r is the distance from the point of application of the force to the pivot, then d = r sinϕ (called the 'lever arm'). We can also write

$$\tau = F_{\perp} r = F \sin\phi\, r$$

where $F_{\perp} = F \sin\phi$ is the component of F that is perpendicular to r.

In vector notation, the torque if the cross product (vector product) of **r** and **F**:

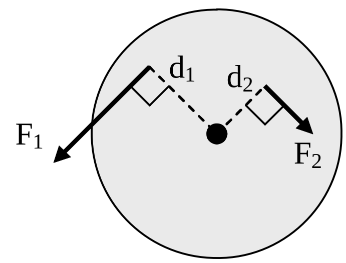
$$\tau = \mathbf{r} \times \mathbf{F}$$

The direction of $\boldsymbol{\tau}$ is perpendicular to the plane formed by **r** and **F**. It would be along the axis of rotation due to the torque.

The torque due to the force shown in the diagram would produce a *counter-clockwise* rotation about the pivot and is assumed to be *positive*. The direction of the torque would be out of the plane. A torque that would produce a *clockwise* rotation is *negative* and would be directed into the plane in the diagram.

<u>Example</u>:

A disk pivoted about an axis through its center is subjected to two forces, as shown to the right. Given: F_1 = 10 N, F_2 = 5 N, d_1 = 8 cm, d_2 = 5 cm. What is the net torque?



$$\tau = F_1 d_1 - F_2 d_2 = (10\text{ N})(0.08\text{ m}) - (5\text{ N})(0.05\text{ m})$$

$$= 0.8\text{ N-m} - 0.25\text{ N-m} = \underline{0.55\text{ N}\cdot\text{m}}$$

Torque and Angular Acceleration

For a single point mass going in a circle subject to a tangential force,

$$F_t = ma_t = mr\alpha \quad \text{(it also has a centripetal component of force and acceleration)}$$

The torque is then

$$F_t r = mr^2\alpha \ ,$$

or

$$\tau = mr^2\alpha$$

For a rigid body consisting of many masses, each mass goes in a circle and has the same angular acceleration α. Thus, the net torque is

$$\tau = \left(\sum mr^2\right)\alpha = I\alpha$$

Example:

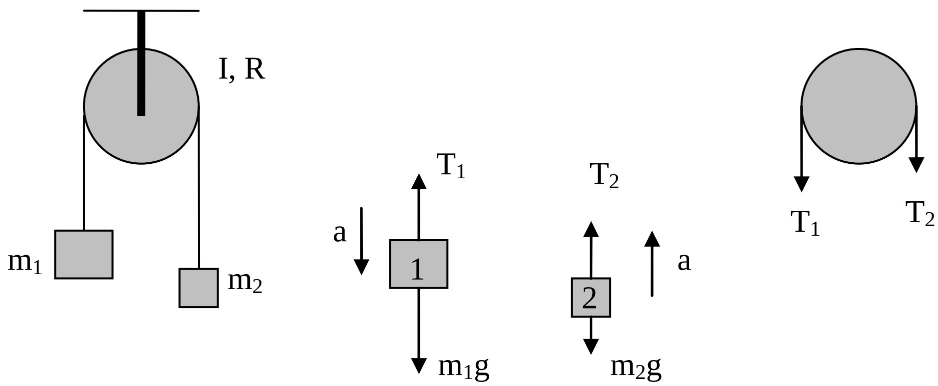
For the disk in the example above that is subjected to two forces, assume that the radius is R = 15 cm and the mass is 5 kg. What is the angular acceleration of the disk?

$$I = \tfrac{1}{2}MR^2 = \tfrac{1}{2}(5kg)(0.15m)^2 = 0.0563 kg \cdot m^2$$

$$\alpha = \frac{\tau}{I} = \frac{0.55 N \cdot m}{0.0563 kg \cdot m^2} = 9.8 rad / s^2$$

Example:

An Atwood machine consists of two masses hanging by a string over a pulley, as shown below.



Determine an expression for the acceleration of the masses, assuming that friction can be neglected. The pulley has inertia I and radius R. We apply the 2nd law to mass 1, mass 2, and the pulley.

$$\begin{aligned}
&mass\ 1: \ m_1 g - T_1 = m_1 a \\
&mass\ 2: T_2 - m_2 g = m_2 a \\
&pulley: \ \sum \tau = T_1 R - T_2 R = I\alpha = Ia / R, \\
&\qquad or,\ T_1 - T_2 = Ia / R^2
\end{aligned}$$

By adding the first two equations we get

$$-T_1 + T_2 + (m_1 - m_2)g = (m_1 + m_2)a$$

Using the third equation to eliminate T_1 - T_2, we get

$$-Ia / R^2 + (m_1 - m_2)g = (m_1 + m_2)a$$

Solving for a,

$$a = \frac{(m_1 - m2)g}{m_1 + m_2 + I / R^2}$$

Work, Power, and Energy

A torque applied to a rigid object can do work on the object if it rotates. The work done in a small angular displacement dθ is

$$dW = \mathbf{F} \cdot d\mathbf{s} = F \sin\phi \, ds = F \sin\phi \, r d\theta \,,$$

where ϕ is the angle between **r** and **F**. Since the τ = Fsinϕr, then

$$dW = \tau \, d\theta$$

The power is the rate at which the work is done in rotating the object and is given by

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau \, \omega$$

The work-energy theorem also applies to the work done by a torque. Since

$$\tau = I\alpha = I \frac{d\omega}{dt} \quad \text{and} \quad d\theta = \omega dt \text{, then}$$

$$W = \int \tau \, d\theta = \int I \frac{d\omega}{dt} \omega \, dt = \int I \omega \, d\omega = \tfrac{1}{2} I\omega^2 - \tfrac{1}{2} I\omega_0{}^2 = \Delta K_{rot}$$

Rolling Objects

A rolling object has both translational and rotational kinetic energy –

$$K = K_{trans} + K_{rot} = \tfrac{1}{2} M v_{cm}^2 + \tfrac{1}{2} I_{cm} \omega^2 ,$$

where v_{cm} is the speed of the center of mass and I_{cm} is the rotational inertia about the rotation axis through the center of mass.

<u>Example</u>:

A ball is released from rest at the top of an inclined plane of height 2 m. What is its speed when it reaches the bottom, assuming it rolls without slipping?

$$E_f = E_i$$

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2 = Mgh$$

$$\omega = v/R$$

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\left(\frac{v}{R^2}\right)^2 = \tfrac{1}{2}\left(M + I/R^2\right)v^2 = Mgh$$

$$v = \sqrt{\frac{2Mgh}{M + I/R^2}}$$

For a solid ball of uniform density, $I = \tfrac{2}{5}MR^2$, so

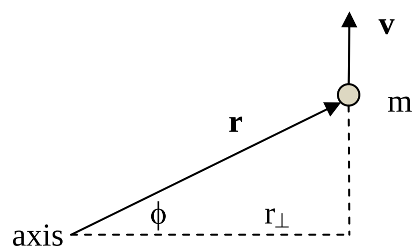
$$v = \sqrt{\frac{2Mgh}{M + \tfrac{2}{5}M}} = \sqrt{\tfrac{10}{7}gh}$$

$$= \sqrt{\tfrac{10}{7}(9.8)(2)} = 5.29\,m/s$$

Angular Momentum

The angular momentum of a moving particle of mass m is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$



The magnitude of **L** is

$$L = mvr\sin\phi = mv_\perp r = mvr_\perp$$

L depends on the distance, $r_\perp$, between the line of motion of the particle and the axis about which **L** is defined.

For a rigid body rotating about a fixed axis, **L** is the sum of the angular momentum of all particles making up the body.

$$L = \sum L_i = \sum m_i v_i r_i = \sum m_i (\omega r_i) r_i = (\sum m_i r_i^2)\omega$$

or,

$$L = I\omega$$

The angular momentum of an object will remain constant unless there is a torque –

$$\frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

But $\frac{d\mathbf{r}}{dt} \times \mathbf{p} = \mathbf{v} \times m\mathbf{v} = 0$ (For any vector, **A** x **A** = 0.)

Thus, $\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \mathbf{F}$, or

$$\tau = \frac{d\mathbf{L}}{dt}$$

This is analogous to Newton's 2nd law relating the force to the time rate of change of linear momentum, $\mathbf{F} = \frac{d\mathbf{p}}{dt}$.

The above equation can be extended to include the total external torque (internal torques involve action-reaction pairs and cancel) and the total angular momentum of a system of particles. This leads to the *law of conservation of angular momentum*, which states that *the total angular momentum of a system of particles is constant if there is no net external torque*.

Example:

A figure skater goes into a spin with her arms outstretched. She then pulls her arms in against her body. If she is initially rotating at 2 rev/s, what is her final rotational speed? Assume that by pulling in her arms she reduces her rotational inertia from 2 kg-m^2 to 1.5 kg-m^2.

$$L_f = L_i$$
$$I_f \omega_f = I_i \omega_i$$
$$\omega_f = \frac{I_i \omega_i}{I_f} = \frac{(2kg \cdot m^2)(2rev/s)}{1.5kg \cdot m^2} = \underline{2.67\, rev/s}$$

How much kinetic energy was gained or lost when she pulled in her arms?

$$KE_f = \tfrac{1}{2} I_f \omega_f{}^2 = \tfrac{1}{2}\left(1.5kg \cdot m^2\right)((2\pi)(2.67)\, rad/s)^2 = 210.5\, J$$
$$KE_i = \tfrac{1}{2} I_f \omega_f{}^2 = \tfrac{1}{2}\left(2kg \cdot m^2\right)((2\pi)(2)\, rad/s)^2 = 157.9\, J$$
$$\Delta KE = KE_f - KE_i = 210.5\, J - 157.9\, J = \underline{52.6\, J}$$

Thus, there was a *gain* in KE. Note that we had to convert angular velocity to rad/s when calculating KE.