Algebra Qualifying Exam, July 2001

Attempt at least **two** *questions from each section. Maximum points can be obtained by answering* **five** *questions correctly, but you may attempt as many questions as you wish. More credit will be given for complete answers than for a number of fragments. All rings are assumed to contain a multiplicative identity.* $\mathbb{N} = \{1, 2, \dots\}$ *and* $\mathbb{Z}$ *denotes the set of integers. For a ring or ideal* R *we let* $M_n(R)$ *denote the set of* $n \times n$ *matrices with coefficients in* R. *All modules will be right modules unless otherwise stated. This exam lasts 4 hours. Good luck!*

Section A

1) Let R be a ring and M an R-module.
(a) Explain what is meant by the term: M is *Noetherian*.

(b) Prove that if M is an R-module and N is an R-submodule such that N and M/N are Noetherian then M is Noetherian.

(c) Prove that M is Noetherian if and only if its submodules are all finitely generated.

(d) Give an example of a ring R that is not Noetherian, giving reasons for your answer.

2) (a) Explain what is meant by a *right Artinian ring*.

(b) Prove in detail that if R is a ring then the ideals of $M_n(R)$ are precisely of the form $M_n(J)$ where J is an ideal of R.
(c) Give an example to show that the statement of (b) is not true if we replace the word "ideal" by "right ideal".
(d) Prove that R is right Artinian if and only if $M_n(R)$ is right Artinian.

3) (a) Let R be a ring. Let A be a right R-module and B a left R-module. Give the definition of the tensor product $A \otimes_R B$. Be careful to define all terms used.

(b) Explain how to construct the tensor product $A \otimes_R B$. Detailed proofs are not required.

(c) Let m, n be natural numbers and let d be the greatest common divisor of m and n. Prove that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$.

4) (a) Recall that a ring R has invariant basis number (IBN) if $R^{(m)} \cong R^{(n)}$ implies $m = n$. It is known that $R^{(m)} \cong R^{(n)}$ if and only if there is an $m \times n$ matrix A and an $n \times m$ matrix B, each with entries in R, such that $AB = I_m$ and $BA = I_n$.

(a) Let $\psi : T \longrightarrow R$ be a ring homomorphism where R has IBN. Prove that T has IBN.

(b) Prove that every commutative ring has IBN.

(c) Prove that $M_k(R)$ has IBN for every $k \in \mathbb{N}$.

(d) Let V be the infinite dimensional vector space over a field F with basis $\{v_1, v_2, \dots\}$ and let $S = \operatorname{End}_F(V) = \hom_F(V, V)$, the endomorphism ring of V. Define $\Phi : S \longrightarrow S^{(2)}$ by defining $\Phi(f) = (f_1, f_2)$ for each $f \in S$ where f_1 and f_2 are given by

$$f_1(v_i) = f(v_{2i}) \quad \text{and} \quad f_2(v_i) = f(v_{2i-1}), \text{for all } i \in \mathbb{N}.$$

Prove that Φ is an isomorphism and deduce that $S^{(m)} \cong S^{(n)}$ for all $m, n \in \mathbb{N}$.

5) (a) Complete the definition: An R-module P is projective if ....

(b) Let X, Y be R-modules. Prove that if there exist R-homomorphisms σ, τ with $\sigma : X \longrightarrow Y, \tau : Y \longrightarrow X$ with $\sigma\tau = 1_Y$ then $X = \ker \sigma \oplus \operatorname{Im} \tau$. Deduce that if P is a projective R-module and $\theta : M \longrightarrow P$ is a surjective R-module homomorphism then $M \cong P \oplus \ker \theta$.

(c) Prove that a free R-module is always projective and give an example of a ring R and a projective R-module which is not free.

6) (a) Let R be a ring. The Jacobson radical of R, denoted by $\operatorname{Rad} R$, is defined to be $\{r \in R | Vr = 0 \text{ for all irreducible } R\text{-modules } V\}$.

(a) Prove that $\operatorname{Rad} R = \cap\{M | M \text{ is a maximal right ideal of } R\}$.

(b) Prove that $\operatorname{Rad} R = \{r \in R | 1 - rs \text{ is right invertible for all } s \in R\}$.

(c) A right ideal A of R is said to have property $(*)$ if it satisfies the following: if B is a right ideal of R and $A + B = R$ then $B = R$. Prove that A has property $(*)$ if and only if $A \leq \operatorname{Rad} R$.

Section B

7) (a) Explain what is meant when we say that the group F is a *free group on a set* X.

(b) Explain how to construct a free group on a set X. Proofs are not required but you should give detailed explanations.

(c) Let n be a fixed natural number and let $G = \langle a, x | a^{2^{n-1}} = 1, x^2 = a^{2n-2}, x^{-1}ax = a^{-1} \rangle$. Find the center, $Z(G)$, the derived subgroup, G', and write a presentation for $G/Z(G)$.

8) (a) State the 3 Sylow theorems.

(b) Prove that if G is a finite group of order p^n, for some prime p then the center of G is non-trivial.

(c) Let p, q, r be (not necessarily distinct) primes. Prove that a group of order pqr has a non-trivial proper normal subgroup.

9) (a) A group G is called *locally cyclic* if every finitely generated subgroup of G is cyclic. Prove that $\mathbb{Q}$, under addition, is an example of a locally cyclic group.

(b) Let p be a prime. Give the definition of the Prüfer group, C_{p^∞} in terms of a system of generators and relations. Give a concrete example of a group that is isomorphic to C_{p^∞}.

(c) Prove that if $G \cong C_{p^\infty}$, for some prime p then G has exactly one subgroup of order p^i for each $i \geq 1$ and that every proper subgroup of G is a finite cyclic group.

10) (a) Let $\theta : H \longrightarrow \operatorname{Aut} N$ be a homomorphism. Explain what is meant by the semidirect product $G = N \rtimes_\theta H$.

(b) Show that there are exactly 5 non-isomorphic groups of order 12.

11) (a) Explain what is meant by a nilpotent group of class n.

(b) Prove that if H is a subgroup of a nilpotent group G then there is a finite series $H_0 = H \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_n = G$ from H to G.

(c) Prove that if G is nilpotent then the set of elements of finite order forms a characteristic subgroup of G.

(d) Show how to construct, for each prime p and each natural number n, a nilpotent p-group of class exactly n. Detailed proofs concerning your assertions are not required.