Algebra Qualifying Exam, July 2007

Attempt at least **two** *questions from each section. Maximum points can be obtained by answering* **five** *questions correctly, but you may attempt as many questions as you wish. More credit will be given for complete answers than for a number of fragments. All rings are assumed to contain a multiplicative identity.* $\mathbb{N} = \{1, 2, \dots\}$ *and* $\mathbb{Z}$ *denotes the set of integers. For a ring or ideal* R *we let* $M_n(R)$ *denote the set of* $n \times n$ *matrices with coefficients in* R*. All modules will be right modules unless otherwise stated. This exam lasts 4 hours. Good luck!*

Section A

1) Let R be a ring. The Jacobson radical of R, denoted by $\operatorname{Rad} R$, is defined to be $\{r \in R | Vr = 0$ for all irreducible R-modules $V\}$.

(a) Prove that $\operatorname{Rad} R = \cap\{M | M$ is a maximal right ideal of $R\}$.

(b) Prove that $\operatorname{Rad} R = \{r \in R | 1 - rs$ is right invertible for all $s \in R\}$.

(c) Determine the Jacobson radical of the ring $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} | a, b, c, d, e, f \in \mathbb{R} \right\}$.

2) 8) (a) Explain what is meant by a *projective module* P over a ring R.

(b) Let A and A' be R-modules. We write $A \sim A'$ if and only if there exist projective R-modules P and P' such that $A \bigoplus P \cong A' \bigoplus P'$. Prove that $\sim$ is an equivalence relation.

(c) Let A be an R-module. Define the projective dimension $\mathrm{pd}_R A$ of A and the global dimension gl dim R of R.

(d) Suppose that $0 \longrightarrow B \longrightarrow P \longrightarrow A \longrightarrow 0$ is exact and that P is projective. Prove that if the projective dimension of A is a positive integer n then the projective dimension of B is $n - 1$. What can be said concerning the projective dimension of B if A has projective dimension 0 or ∞?

3) (a) State the Artin-Wedderburn theorem and then write a short essay indicating some of the main steps and ideas in its proof.

(b) If R is a ring then the *center*, $Z(R)$, of R is defined to be $\{r \in R | rs = sr \text{ for all } s \in R\}$. Prove that if F is a field then $Z(M_n(F)) = \{rI | r \in F\}$, where I is the $n \times n$ identity matrix.

(c) Prove that the center of a Wedderburn ring is a direct sum of fields.

4) (a)Let A be a simple R-module. Prove that $\text{Hom}_R(A, A)$ is a division ring.

(b) Let R be a ring. Prove that the ideals of $M_2(R)$ are precisely of the form $M_2(J)$, where J is an ideal of R. Deduce that if R is a field then $M_2(R)$ is a simple ring.

(c) Construct a simple ring that contains exactly 2401 elements.

5) (a) Let R be a ring. Explain what is meant when we say that the R-module M is Artinian.

(b) Prove that if R is a ring, M is an R-module and N is an R-submodule of M then M is Artinian if and only if $N, M/N$ are both Artinian.

(c) Prove that if R is a right Artinian ring and M is a finitely generated R-module then M is also Artinian. Hence prove that if a ring R is Artinian then for each natural number n the full matrix ring $M_n(R)$ is also Artinian.

6) Recall that a ring R has invariant basis number (IBN) if $R^{(m)} \cong R^{(n)}$ implies $m = n$.

(a) Give a condition in terms of matrices which ensures that $R^{(m)} \cong R^{(n)}$.

(b) Let $\psi : T \longrightarrow R$ be a ring homomorphism where R has IBN. Prove that T has IBN. Deduce that every commutative ring has IBN.

(c) Prove that $M_k(R)$ has IBN for every $k \in \mathbb{N}$ if R is a ring with IBN and hence give an example of a noncommutative ring with IBN.

(d) Prove that every Artinian ring has IBN.

Section B

7) (a) State the three Isomorphism theorems (the first one is sometimes called the Homomorphism Theorem). Prove **one** of these theorems.

(b) An abelian group A is called *divisible* if for each $n \in \mathbb{Z} \setminus \{0\}$ the homomorphism $\theta_n : A \longrightarrow A$ defined by $\theta_n(a) = na$ is onto (or equivalently for each $n \in \mathbb{Z} \setminus \{0\}, a \in A$, $nx = a$ has at least one solution $x \in A$). Prove: Every nontrivial divisible abelian group is infinite.

(c) Give an example of a nontrivial divisible abelian torsion group G and prove in detail that G is divisible.

8) (a) State the three Sylow theorems.

(b) Let G be a group of order pq, where p, q are primes such that $p < q$. Let S_q be a Sylow q-subgroup of G. Prove that S_q is normal in G.

(c) Let $q \not\equiv 1 \pmod{p}$ and let S_p be a Sylow p-subgroup. Prove that S_p is also normal in G.

9) (a) Explain what is meant by *a composition series* and *a normal series* for a group. Explain what is meant when we say that a group is *solvable* (or soluble).

(b) State Schreier's Refinement theorem and the Jordan-Hölder theorem.

(c) Prove in detail that a solvable group having a composition series must be finite.

11) (a) State and prove Lagrange's theorem.

(b) Let G be a finite group of order mn where m, n are relatively prime. For any subgroup H of G let $N_G(H) = \{x \in G | xHx^{-1} = H\}$. Clearly $N_G(H)$ is a subgroup of G and $H \triangleleft N_G(H)$. Prove in detail that if $H \leq G$ is of order m then the number of subgroups conjugate to H in G divides n.

12) Let G be a group and H a subgroup of G. Define $C_G(H) = \{x \in G | hx = xh \text{ for all } h \in H\}$ and $N_G(H) = \{x \in G | xHx^{-1} = H\}$. Prove in detail that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group of H. State the theorems you are using.