Algebra Qualifying Exam, July 2008

Attempt at least **two** *questions from each section. Maximum points can be obtained by answering* **five** *questions correctly, but you may attempt as many questions as you wish. More credit will be given for complete answers than for a number of fragments. All rings are assumed to contain a multiplicative identity.* $\mathbb{N} = \{1, 2, \dots\}$ *and* $\mathbb{Z}$ *denotes the set of integers. For a ring or ideal* R *we let* $M_n(R)$ *denote the set of* $n \times n$ *matrices with coefficients in* R*. All modules will be right modules unless otherwise stated. This exam lasts 4 hours. Good luck!*

Section A

1) Let G be a (not necessarily finite) group and R a commutative ring.

(a) Explain what is meant by the group ring RG.

(b) Suppose that G is finite and that R is a field. Prove that RG is Artinian.

(c) Explain why $\mathbb{Q}G$ is Wedderburn if G is finite. (A detailed proof is not required.)

(d) Let p be a prime and G a finite abelian p-group. Prove that $\mathbb{F}_pG$ is not Wedderburn. (Hint: consider an ideal of the form $(1-x)\mathbb{F}_pG$ where $1 \neq x \in G$.)

(e) Prove that $\mathbb{F}_pC_{p^\infty}$ has a nil ideal that is not nilpotent.

2) (a) Explain what is meant by a *projective module* P over a ring R.

(b) Let A and A' be R-modules. We write $A \sim A'$ if and only if there exist projective R-modules P and P' such that $A \bigoplus P \cong A' \bigoplus P'$. Prove that $\sim$ is an equivalence relation.

(c) Let A be an R-module. Define the projective dimension $\mathrm{pd}_R A$ of A and the global dimension $\mathrm{gl\ dim}\ R$ of R.

(d) Suppose that $0 \longrightarrow B \longrightarrow P \longrightarrow A \longrightarrow 0$ is exact and that P is projective. Prove that if the projective dimension of A is a positive integer n then the projective dimension of B is $n-1$. What can be said concerning the projective dimension of B if A has projective dimension 0 or ∞?

3) Let R be a ring.

(a) Explain what is meant by the expressions 'R is a Wedderburn ring' and 'R is an Artinian ring'. Write a short essay outlining the basic theory of projective modules over Artinian rings.

(b) Suppose R is Artinian and $R/Nil(R)$ is a simple ring. How many simple R-modules are there (up to isomomophism)? How many indecomposable projective R-modules? How many projective R-modules?

4) (a)Let A be a simple R-module. Prove that $\mathrm{Hom}_R(A, A)$ is a division ring.

(b) Give an example of a division ring that is not a field.

(c) Prove directly that if D is a division ring then $M_2(D)$ is a simple ring.

(d) Construct a simple ring that contains exactly 256 elements.

5) (a) Let R be a ring. Explain what is meant when we say that the R-module M is Artinian.

(b) Prove that if R is a ring, M is an R-module and N is an R-submodule of M then M is Artinian if and only if $N, M/N$ are both Artinian.

(c) Prove that if R is a right Artinian ring and M is a finitely generated R-module then M has a composition series. You may assume that M is Artinian. (Hint: Consider the series $M \geq MN \geq MN^2 \geq MN^3 \ldots$ where $N = Nil(R)$.)

6) Let R be a ring and G a non-trivial finite group.

(a) Let $e \in R$ be an idempotent. Show that $R = eR \oplus (e-1)R$ as right R-modules.

(b) Let $\sigma = \Sigma_{g \in G} g \in \mathbb{Q}G$. Compute σ^2 and deduce that $\mathbb{Q}G$ contains a non-trivial idempotent.

(c) Show that $\mathbb{Q}G$ is not a simple right $\mathbb{Q}G$-module.

(d) Suppose that G is abelian and that $\mathbb{Q}G$, viewed as a $\mathbb{Q}G$-module, is a direct sum of exactly two simple modules. What can be said about the structure of G?

(e) Let H be an infinite group. Can $\mathbb{Q}H$ be simple as a right $\mathbb{Q}H$-module?

Section B

7) (a) Explain what is meant by a divisible group.

(b) Prove that every non-trivial divisible group is infinite and that if G is divisible and $N \triangleleft G$ then G/N is also divisible..

(c) Give an example of a non-trivial periodic divisible group, being sure to explain why your example is divisible.

(d) State the main structure theorems for finitely generated abelian groups and deduce that if G is a non-trivial finitely generated abelian group then G cannot be divisible.

8) (a) State the three Sylow theorems.

(b) Suppose that G is a group and N is a normal subgroup of G. Let

$$C_G(N) = \{g \in G | gx = xg \text{ for all } x \in G\}.$$

Prove that $G/C_G(N)$ is isomorphic to a subgroup of the automorphism group of N.

(c) Show that if G is a cyclic group of prime order q then $\mathrm{Aut}\,(G)$ has exactly $q-1$ elements. Use the results of (a) and (b) together with this fact to prove: If $p < q$ are distinct primes then there exists a non-abelian group of order pq if and only if p divides $q-1$.

9) (a) State Lagrange's Theorem.

(b) Show that A_4 has a subgroup of index 3.

(c) Let G be a group and let $H \leq G$ be a subgroup such that $|G : H| = n < \infty$. Let $X = \{gH \mid g \in G\}$ be the set of left H-cosets. Prove **in detail** that there is a homomorphism $\Phi : G \longrightarrow \mathrm{Sym}\, X$ and that if N is the kernel of this map then $|G : N| \leq n!$.

(d) Prove that if $n \geq 5$ then A_n has no proper, nontrivial subgroup of index strictly less than n.

10) (a) Let $\theta : H \longrightarrow \operatorname{Aut} N$ be a homomorphism. Explain what is meant by the semidirect product $G = N \rtimes_\theta H$.

(b) Let $D = \{a/b \in \mathbb{Q} | a \in \mathbb{Z}, b = 2^n \text{ for some } n \in \{0\} \cup \mathbb{N}\}$, an **additive** subgroup of the rational numbers $\mathbb{Q}$. Prove that there is an automorphism α of D of infinite order given by $\alpha(d) = d/2$ for all $d \in D$.

(c) Form the semidirect product $G = D \rtimes \langle \alpha \rangle$. Compute $Z(G)$ and G' and show that every nontrivial element of G has infinite order.

(d) Prove that G is soluble, but not nilpotent.

11) (a) Explain what is meant by a soluble (solvable) group.

(b) Let G be a group and let $N \lhd G$. Prove that G is soluble if and only if both N and G/N are soluble. Show also that if M, N are soluble normal subgroups of the group G then MN is also soluble.

(c) Let G be a group and let H, K be normal subgroups of G such that $H \cap K = 1$. Prove that if G/H and G/K are both soluble then G is also soluble.

(d) Give an example of a non-soluble group. Explain why your example works.

12) (a) Explain what is meant by a nilpotent group and what is meant by the nilpotency class of a nilpotent group.

(b) Let p be a prime and let $n \in \mathbb{N}$. Prove that if G is a finite group of order p^n then $Z(G) \neq 1$ and use this to deduce that a group of order p^n is always nilpotent.

(c) Prove that a finite direct product of nilpotent groups is nilpotent.

(d) Is part (c) still true if the word "finite" is removed? Give a proof or a counterexample.