Algebra Qualifying Exam, June 1999

Attempt at least **two** *questions from each section. Maximum points can be obtained by answering* **five** *questions correctly, but you may attempt as many questions as you wish. More credit will be given for complete answers than for a number of fragments. All rings are assumed to contain a multiplicative identity.* $\mathbb{N} = \{1, 2, \dots\}$ *and* $\mathbb{Z}$ *denotes the set of integers.*

Section A

1) Complete the definition: A group G is nilpotent of class at most c if ....

(b) Let G be a finite group. Prove that G is nilpotent if and only if it is the direct product of its Sylow p-subgroups.

(c) Let p be a prime and c a natural number. Construct a finite p-group that is nilpotent of class c. Construct an infinite non-nilpotent group which is the direct product of its maximal p-subgroups.

2) (a) State the 3 Sylow theorems.

(b) Let G be a group of order p^2q^2, where p, q are primes, $p < q$ and $p \nmid (q^2 - 1)$. Prove that $G = P \times Q$, where $|P| = p^2$ and $|Q| = q^2$.

(c) Prove that if p is a prime then a finite p-group has non-trivial center and deduce that the group G in part (b) is abelian.

3) Suppose G is a group and $H \leq G$. Let $X = \{Hg : g \in G\}$ be the set of right H-cosets.

(a) Prove that there is a homomorphism from G into $\operatorname{Sym} X$, the symmetric group on X.

(b) Prove that if $n \in \mathbb{N}$ and $|G : H| = n$ then G has a normal subgroup K, contained in H such that $|G : K| \leq n!$

(c) Deduce that if G is a finite group and p is the smallest prime dividing the order of G then any subgroup H of index p in G is normal in G.

4) A group G is called *periodic* if every element of G has finite order and G is called *torsion-free* if every non-trivial element of G has infinite order.

(a) Let $N \lhd G$. Prove that G is periodic if and only if N and G/N are both periodic.

(b) Let I be an index set and, for each $i \in I$, let $N_i \lhd G$ be a periodic normal subgroup of G. Prove that the product $\prod_{i \in I} N_i$ is also periodic.

(c) Prove that in an abelian group the set of elements of finite order in G forms a characteristic subgroup of G.

(d) Show, by constructing examples, that (a) and (b) are false if we replace the word "periodic" by "torsion-free".

5) (a) Let $\theta : H \longrightarrow \operatorname{Aut} N$ be a homomorphism. Explain what is meant by the semidirect product $G = N \rtimes_\theta H$.

(b) Let $D = \{a/b \in \mathbb{Q} | b = 2^n \text{ for some } n \in \{0\} \cup \mathbb{N}\}$, an additive subgroup of the rational numbers $\mathbb{Q}$. Prove that there is an automorphism α of D given by $\alpha(d) = d/2$ for all $d \in D$ and that $\theta : C_\infty \longrightarrow \operatorname{Aut} D$ given by $\theta(x) = \alpha$ is a homomorphism. Here C_∞ is the infinite cyclic group generated by x.

(c) Let $G = D \rtimes_\theta \langle \alpha \rangle$. Show that G can be generated by two elements. Deduce that subgroups of finitely generated groups need not be finitely generated.

6) Let $Z(G)$ and $\operatorname{Inn} G$ denote the center of the group G and the group of inner automorphisms of G respectively.

(a) Show that $G/Z(G) \cong \operatorname{Inn} G$.

(b) Suppose that $\operatorname{Aut} G$ is cyclic. Show that G is abelian.

(c) Describe the automorphism group of S_3, the symmetric group on three symbols.

Section B

7) (a) Let R be a ring. Complete the definition: The R-module M is Artinian if ....

(b) Prove that if R is a ring, M is an R-module and N is an R-submodule of M then M is Artinian if and only if $N, M/N$ are both Artinian.

(c) Prove that if R is a right Artinian ring and M is a finitely generated R-module then M is also Artinian. Deduce that if R is Artinian then the full matrix ring $M_n(R)$ is also Artinian.

8) Let R be a ring. The Jacobson radical $\operatorname{Rad} R$ of R is defined to be $\{r \in R | Vr = 0 \text{ for all irreducible } R\text{-modules } V\}$.

(a) Prove that $\operatorname{Rad} R = \cap\{M | M \text{ is a maximal right ideal of } R\}$.

(b) Prove that $\operatorname{Rad} R = \{r \in R | 1 - rs \text{ is right invertible for all } s \in R\}$.

(c) If R_1, R_2 are rings and $R = R_1 \oplus R_2$ then prove that $\operatorname{Rad} R = \operatorname{Rad} R_1 \oplus \operatorname{Rad} R_2$.

9) (a) Let R be a ring. Complete the following definition: The R-module M is completely reducible ....

(b) Prove that if I is a right ideal of the ring R then I is a direct summand of R if and only if $I = eR$ for some idempotent e of R.

(c) Prove the theorem that if R_R is completely reducible then R is Artinian and contains no non-zero nilpotent ideals. Give an example of a ring R such that R_R is not completely reducible.

10) (a) Complete the definition: An R-module P is projective if ....

(b) Let X, Y be R-modules. Prove that if there exist R-homomorphisms σ, τ with $\sigma : X \longrightarrow Y, \tau : Y \longrightarrow X$ with $\sigma\tau = 1_Y$ then $X = \ker\sigma \oplus \operatorname{Im}\tau$. Deduce that if P is a projective R-module and $\theta : M \longrightarrow P$ is a surjective R-module homomorphism then $M \cong P \oplus \ker\theta$.

(c) Prove that a free R-module is always projective and give an example of a ring R and a projective R-module which is not free.

11) Recall that a ring R has invariant basis number (IBN) if $R^{(m)} \cong R^{(n)}$ implies $m = n$. It is known that $R^{(m)} \cong R^{(n)}$ if and only if there is an $m \times n$ matrix A and an $n \times m$ matrix B, each with entries in R, such that $AB = I_m$ and $BA = I_n$.

(a) Let $\psi : T \longrightarrow R$ be a ring homomorphism where R has IBN. Prove that T has IBN. Deduce that every commutative ring has IBN.

(b) Prove that if R has IBN then $M_k(R)$ has IBN for every $k \in \mathbb{N}$.

(c) Let V be the infinite dimensional vector space over a field F with basis $\{v_1, v_2, \dots\}$ and let $S = \operatorname{End}_F(V) = \hom_F(V, V)$, the endomorphism ring of V. Define $\Phi : S \longrightarrow S^{(2)}$ by defining $\Phi(f) = (f_1, f_2)$ for each $f \in S$ where f_1 and f_2 are given by

$$f_1(v_i) = f(v_{2i}) \quad \text{and} \quad f_2(v_i) = f(v_{2i-1}), \text{for all } i \in \mathbb{N}.$$

Prove that Φ is an isomorphism and deduce that $S^{(m)} \cong S^{(n)}$ for all $m, n \in \mathbb{N}$.