Module #5

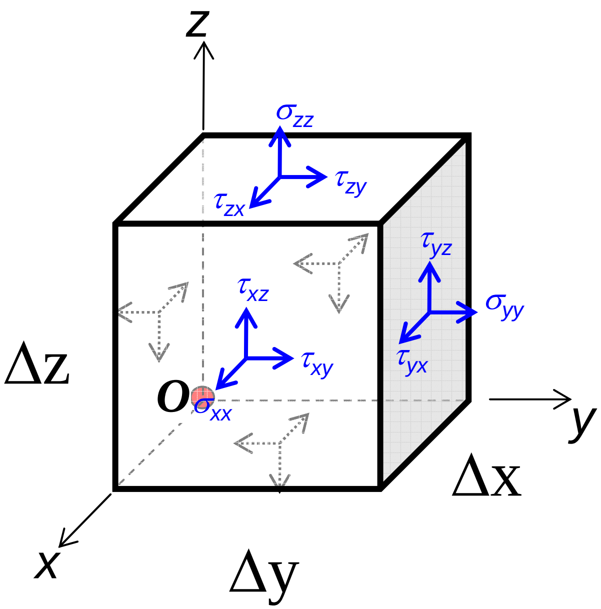
Stress at a point

Tensors

Transformation of stresses

REFERENCE

Ch. 2, Pages 17-24 and 30-35 in Dieter



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- Draw the planes that are perpendicular to each face. This defines a small cubic (or parallelepiped-shaped) volume element.

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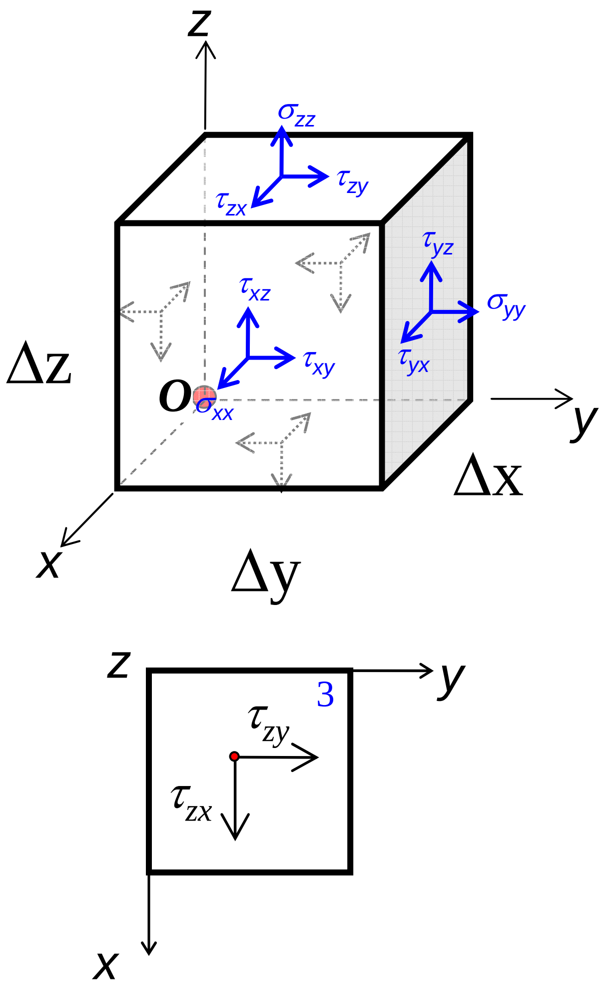


- For each plane, we can now define one normal stress and two shear stresses. *This constitutes the state of stress.*

- This construction automatically implies the need for 18 stress components to define the state of stress in the volume element.

Recall from statics

At equilibrium, ΣF = 0 & ΣM = 0

(there can be no net force or torque)

Sum of Forces Parallel To Each Axis:

$$x\text{-axis: } \sigma_{xx}A-\sigma_{-x-x}A=0\Rightarrow\sigma_{xx}=\sigma_{-x-x}$$

$$y\text{-axis: } \sigma_{yy}A-\sigma_{-y-y}A=0\Rightarrow\sigma_{yy}=\sigma_{-y-y}$$

$$z\text{-axis: } \sigma_{zz}A-\sigma_{-z-z}A=0\Rightarrow\sigma_{zz}=\sigma_{-z-z}$$

Sum of Moments About Each Axis:

$$z\text{-axis: } \left(\tau_{xy}\Delta y\Delta z\right)\Delta x=\left(\tau_{yx}\Delta x\Delta z\right)\Delta y\Rightarrow\tau_{xy}=\tau_{yx}$$

$$y\text{-axis: } \left(\tau_{xz}\Delta z\Delta y\right)\Delta x=\left(\tau_{zx}\Delta x\Delta y\right)\Delta z\Rightarrow\tau_{xz}=\tau_{zx}$$

$$x\text{-axis: } \left(\tau_{zy}\Delta y\Delta x\right)\Delta z=\left(\tau_{yz}\Delta x\Delta z\right)\Delta y\Rightarrow\tau_{zy}=\tau_{yz}$$

18

$\boxed{-3}$ =15

15

$\boxed{-3}$ =12

SUMMATION OF FORCES ON EACH FACE

These relationships allow us to reduce the number of stress components that must be specified to define the state of stress.

(on *x*-faces):

$$\sigma_{xx} dydz - \sigma_{-x-x} dydz = 0 \Rightarrow \boxed{\sigma_{xx} = \sigma_{-x-x}}$$

$$\tau_{xy} dydz - \tau_{-x-y} dydz = 0 \Rightarrow \boxed{\tau_{xy} = \tau_{-x-y}}$$

$$\tau_{xz} dydz - \tau_{-x-z} dydz = 0 \Rightarrow \boxed{\tau_{xz} = \tau_{-x-z}}$$

☑ 12

-2 =10

(on *y*-faces):

$$\sigma_{yy} dxdz - \sigma_{-y-y} dxdz = 0 \Rightarrow \boxed{\sigma_{yy} = \sigma_{-y-y}}$$

$$\tau_{yx} dxdz - \tau_{-y-x} dxdz = 0 \Rightarrow \boxed{\tau_{yx} = \tau_{-y-x}}$$

$$\tau_{yz} dxdz - \tau_{-y-z} dxdz = 0 \Rightarrow \boxed{\tau_{yz} = \tau_{-y-z}}$$

☑ 10

-2 = 8

(on *z*-faces):

$$\sigma_{zz} dxdy - \sigma_{-z-z} dxdy = 0 \Rightarrow \boxed{\sigma_{zz} = \sigma_{-z-z}}$$

$$\tau_{zx} dxdy - \tau_{-z-x} dxdy = 0 \Rightarrow \boxed{\tau_{zx} = \tau_{-z-x}}$$

$$\tau_{zy} dxdy - \tau_{-z-y} dxdy = 0 \Rightarrow \boxed{\tau_{zy} = \tau_{-z-y}}$$

☑ 8

-2 = **6**

[18 - 12 = 6 components]

Stress Tensor

We can now define the complete state of stress at a point in terms of six components of stress

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \text{ or } \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \text{or } \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \cdot & \sigma_{yy} & \tau_{yz} \\ \cdot & \cdot & \sigma_{zz} \end{bmatrix}$$

$$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{xz}$$

A TENSOR IS

a group of numbers that represents a physical quantity.

Tensors have special properties that makes it easy to transform and manipulate physical quantities.

Tensor Notation

- The rank* of a tensor determines the number of tensor components.
  - Number of components = 3^n where n = rank.
- The rank also determines the number of direction cosines required to transform that physical quantity from one coordinate system to another. More on this later…
  - Number of direction cosines = n where n = rank

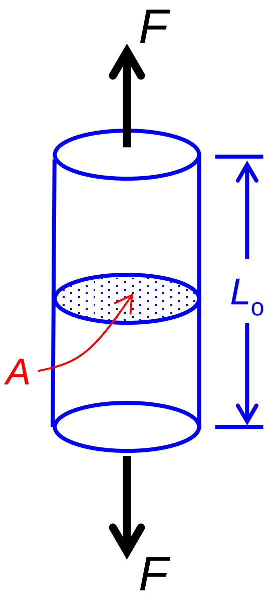
* The rank is also known as the order of the tensor

Tensor Notation

- Ranking of tensors:
  - Zero rank – *scalars*. Scalars are non-directional.
    - Temperature
  - First rank tensors – *vectors*. Vectors are directional.
    - Force
    - Area
  - Second rank tensors – what you get when you relate two vectors to each other.
    - Stress (force & area)
    - Strain (displacement & displacement)
  - Fourth rank tensors – What you get when you relate two second rank tensors.
    - Elastic modulus (stress & strain)

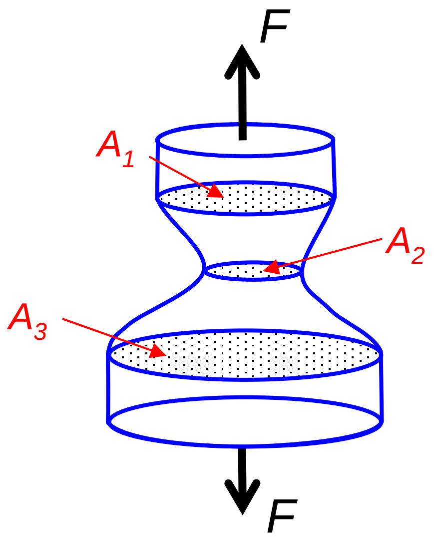
RECALL

The stress state (and strain state) depends on the area of interest and/or the direction of loading relative to that area.



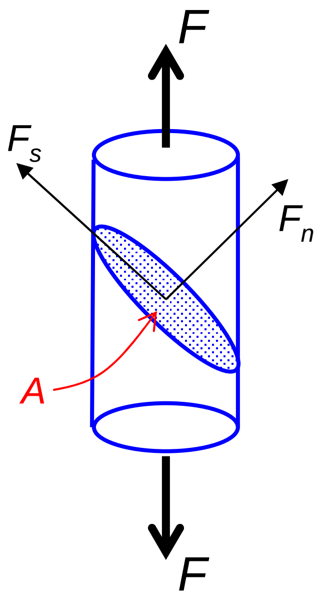
A

Round rod in tension.
Stress distribution is
uniform throughout
the structure.



B

Non-uniform structure in tension.
Stress distribution varies with
position.



C

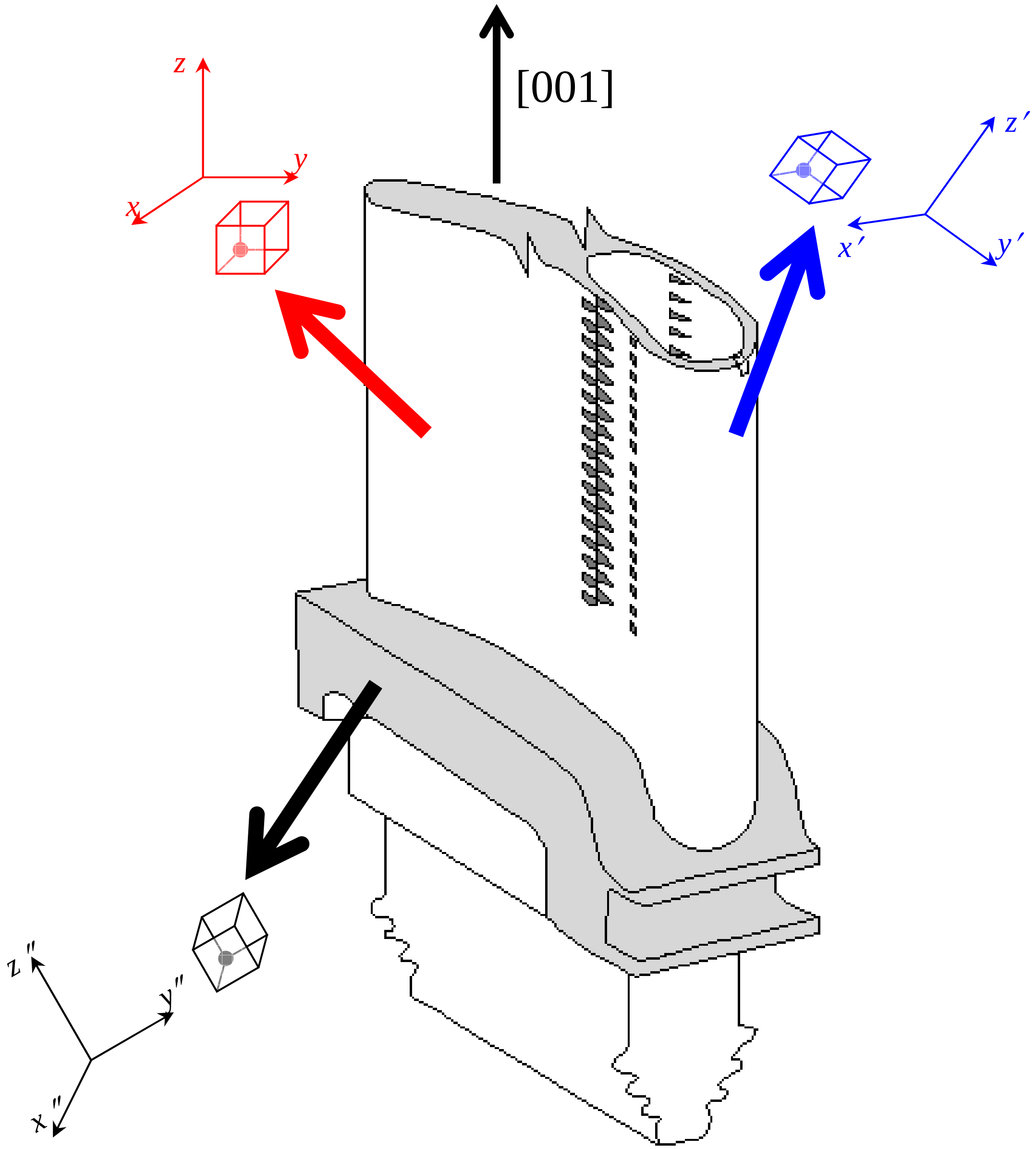
Area of interest not
perpendicular to load.
Stress can be re-defined
relative to a preferred
coordinate system.

Shear stress: $\tau = F_s/A$
Normal stress: $\sigma = F_n/A$

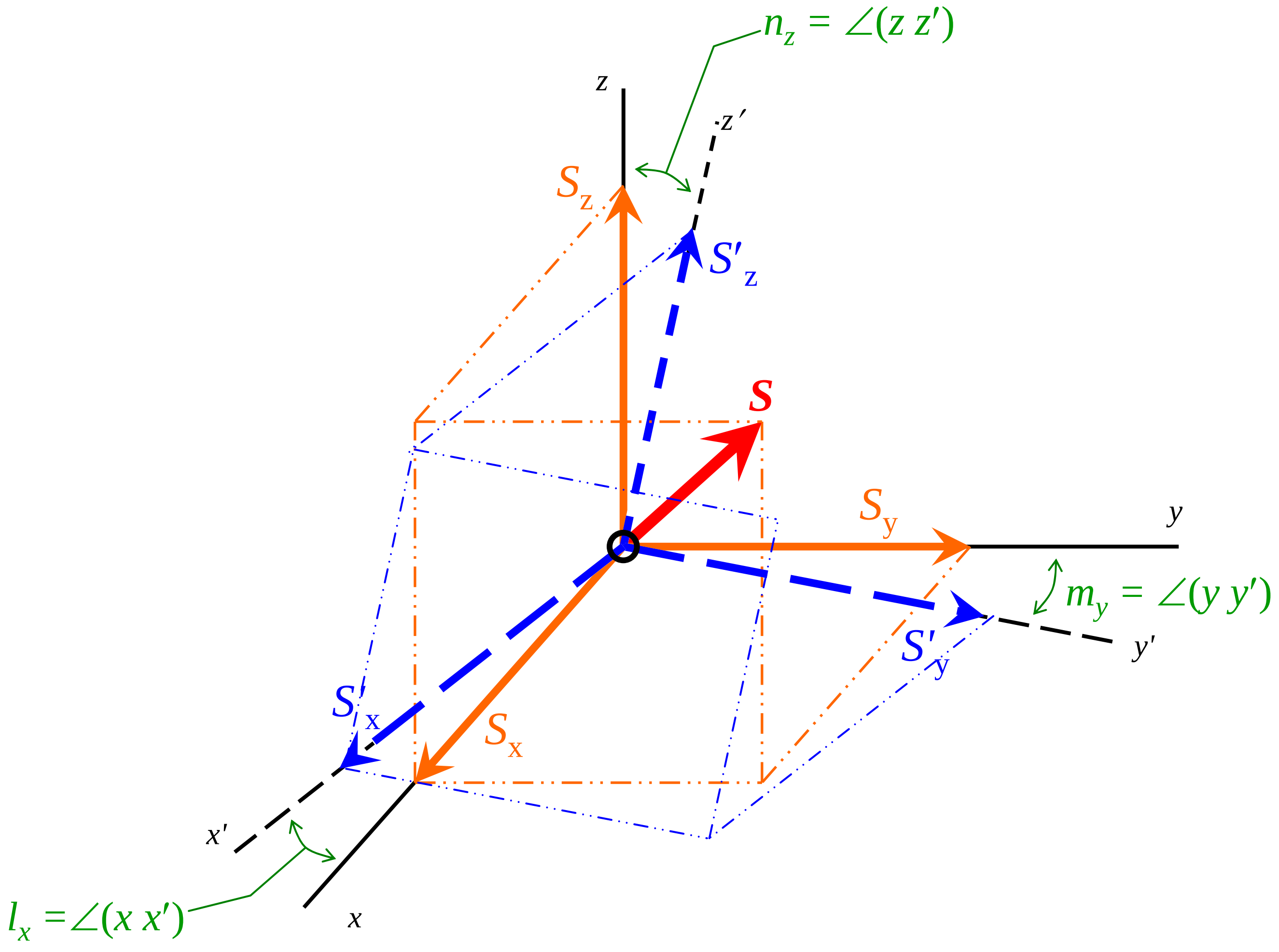
Nearly all solid objects have non-uniform structures (microscopically and macroscopically).

THUS THE STATES OF STRESS GENERALLY VARY FROM POINT TO POINT EVEN THOUGH THE APPLIED FORCES DO NOT CHANGE.

Sometimes it is necessary and/or convenient to express the stresses and strains relative to different coordinate systems.



Transformation of Vectors



THE VECTOR ***S*** CAN BE EASILY RESOLVED INTO COMPONENTS THAT ARE PARALLEL TO ANY SET OF REFERENCE AXES.

$$\overbrace{S_i}^{\text{Vector } \boldsymbol{S} \text{ resolved onto } xyz}$$

$$\left.\begin{array}{c}\text{Vector } \boldsymbol{S}\\ \text{resolved onto}\\ x'y'z'\end{array}\right\} S_i'$$

	x	y	z
x'	$\cos(\angle xx')$	$\cos(\angle yx')$	$\cos(\angle zx')$
y'	$\cos(\angle xy')$	$\cos(\angle yy')$	$\cos(\angle zy')$
z'	$\cos(\angle xz')$	$\cos(\angle yz')$	$\cos(\angle zz')$

OR

S_i

S_i'

	x	y	z
x'	l_x	l_y	l_z
y'	m_x	m_y	m_z
z'	n_x	n_y	n_z

All components are related to each other through a series of “direction cosines”

$$\begin{bmatrix} S'_x \\ S'_y \\ S'_z \end{bmatrix} = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_z & n_z \end{bmatrix} \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix}$$

or

$$S'_i = a_{ij} S_j$$

Vector ***S*** transformed to *x′,y′,z′*

Direction cosines relating *x,y,z* to *x′,y′,z′*

Original vector ***S*** resolved onto *x,y,z*

Since second rank tensors denote relationships between two vectors (*A* and *F*), we require two rotation matrices (i.e., two sets of direction cosines)
to
re-orient them.

Stress and strain are second rank tensors since they require that we specify the directions of plane normals and applied forces.

$$\boxed{\sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl}}$$

2 direction cosines
to transform

The transformation equations denoted above can be applied to any first or second rank tensor. Similar relationships exist for higher rank (order) tensors.

You can see Nye's classic text or one of the other references below for more details.*

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*J.F. Nye: *Physical Properties of Crystals*, Oxford University Press, Oxford, 1957, 1985
A. Kelly, G.W. Groves, & P. Kidd: *Crystallography and Crystal Defects*, John Wiley & Sons, New York, 2000
D.R. Lovett: *Tensor Properties of Crystals*, 2nd ed., IOP Publishing, Philadelphia, 1999

Another important thing to note about second rank tensors:

if they are symmetric it is possible to define a unique set of axes where the tensor will have no off-diagonal components.

$$T_{ij} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} \Rightarrow T'_{ij} = \begin{bmatrix} T_{x'x'} & 0 & 0 \\ 0 & T_{y'y'} & 0 \\ 0 & 0 & T_{z'z'} \end{bmatrix}$$

The new coordinate axes are called the ***principal axes*** and the tensor components related to them are called the ***principal tensor components***.

They represent the most extreme values of the tensor quantity in the system

(i.e., the max. and min. values of the tensor quantity).

Closing Remark

PRINCIPAL STRESSES (& PRINCIPAL STRAINS) ALLOW US TO DETERMINE THE MAXIMUM STRESSES (& STRAINS) AT ANY POINT IN A SOLID.
(*This is important!*)

THEY WILL ALLOW US TO DEFINE YIELD AND FAILURE CRITERIA (more on this later).

Next let's illustrate the concept of principal stresses for a common stress state.