INTERPLANETARY TRANSFERS
THE PATCHED CONIC APPROXIMATION

Introduction

A satellite spends the majority of its interplanetary journey under the primary influence of one body – the sun. Only when very close to the original and target planets is the motion of the satellite under the primary influence of any other body in the solar system. Therefore, as a first approximation when designing interplanetary transfers, the patched conic method can be used. This method involves dividing the transfer into three phases, each phase depending on which body in the solar system is having the primary effect on the satellite's motion. For example, one can surely conclude that the earth is the primary body of a satellite in an orbit only a few hundred kilometers above the surface, such as the space shuttle. However, once a satellite gets far enough from Earth, the sun is the primary body, with the earth's effect becoming insignificant. During most of the journey of the satellite, the sun is the primary body. Finally, the satellite approaches the target planet, and as it gets closer, the target planet's gravitational pull takes over – it becomes the primary body.

In the patched conic method, the three phases of the orbit, corresponding to the three primary bodies, are examined by looking at the orbit around each primary body in turn, discarding all other bodies. For example, a transfer from Earth to Mars (illustrated in **Fig. 1**) involves a hyperbolic escape from Earth, an elliptical transfer about the sun, and a hyperbolic (but capture by applying a ΔV) orbit about Mars. Each phase can be examined independently. It is convenient to begin by examining the sun-primary phase, which is the "middle" part of the transfer.

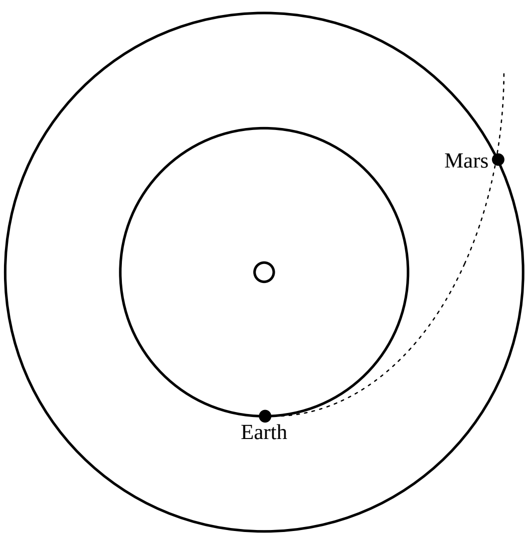

Fig. 1. Semi-tangential Transfer from Earth to Mars. The position of Earth is shown at launch, while the position of Mars is shown at arrival.

The Sun-Primary Phase

While this is the second part of the transfer, it is convenient to begin the calculations here, since the required velocities for this orbit will dictate the velocity of the satellite when it leaves the influence of the earth. For now, let us assume that the influence of the earth is so small compared to the size of the solar system that, for all practical purposes, we can ignore the Earth, and consider our satellite leaving an orbit equivalent to Earth's, and orbiting the sun. (Later, when we determine how far into the solar system the earth's influence extends, you will see that this is a valid assumption.) We have already done all of these calculations. This phase of the patched conic method is merely a semi-tangential transfer from an orbit about the sun with a radius of 1 a.u. to an orbit with the same radius as the orbit of Mars, as illustrated in **Fig. 2**.

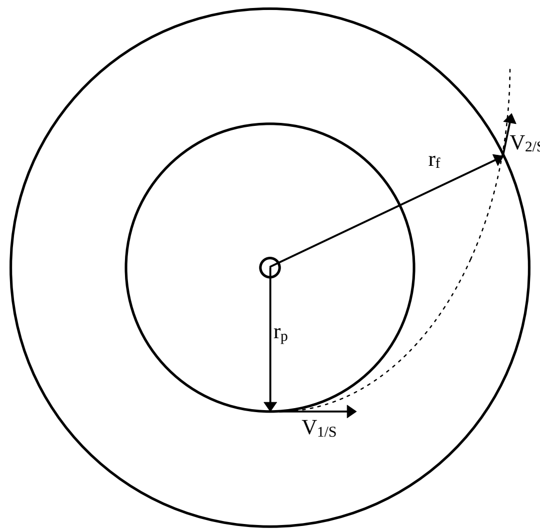

Fig. 2. The Sun-Primary Phase of a Semi-Tangential Transfer from Earth to Mars. $V_{1/S}$ should be read "velocity at point 1 relative to the sun."

$V_{1/S}$ should be read as "velocity of the satellite at the beginning of the transfer relative to the sun." Here is a review of the steps necessary to determine $V_{1/S}$. Note that this can be an iterative process!

1. Apply a ΔV to make the velocity of the satellite greater than that required to continue in a circular orbit about the sun and also to make the velocity greater than required for a Hohmann transfer.

2. Knowing this velocity ($V_{1/S}$) allows the calculation of the semi-major axis (a) from the *vis-viva* equation.

3. Calculate the eccentricity from

$$r_p = a(1-e)$$

4. Obtain the true anomaly (ν) from

$$r_f = \frac{a\left(1-e^2\right)}{1+e\cos(\nu)}$$

5. Calculate the eccentric anomaly:

$$\tan\left(\frac{\nu}{2}\right)=\left(\frac{1+e}{1-e}\right)^{1/2}\tan\left(\frac{E}{2}\right)$$

6. Calculate the mean anomaly. Remember! E must be in radians!

$$M = E - e\sin E$$

7. Calculate the mean motion:

$$n=\sqrt{\frac{\mu}{a^3}}$$

8. Now, the time to transfer is

$$t=\frac{M}{n}$$

9. The phase angle (the angle between Earth and Mars, with the sun being the vertex) required for mission success is

$$\theta=\nu-\frac{t}{P_f}2\pi$$

where P_f is the period of the target orbit, and all angles are radians.

10. Wait time can be calculated the same as for a Hohmann transfer.

Note that it may not be necessary to do all of the above calculations. Also, if the phase angle or transfer time is unacceptable, and new ΔV can be chosen. In addition, it may not be possible to choose a ΔV since this obviously depends on the propulsion system on the spacecraft.

After doing certain steps above, the mission designer has enough information to determine the requirements for Earth departure, i.e., when the satellite leaves the influence of the earth.

Sphere of Influence

When does the satellite leave the influence of the earth and become solely under the influence of the sun. Obviously, that cannot happen in reality – suddenly turning off Earth's gravity and turning on the sun's – but that is the way the patched conic approximation works. The sphere of influence of a planet can be calculated from the following equation:

$$R_{SOI}=a_{planet}\left(\frac{\mu_{planet}}{\mu_{sun}}\right)^{2/5}$$

where R_{SOI} is the radius of the sphere of influence, and a_{planet} is the semi-major axis of the planet's orbit about the sun. Therefore, when the satellite reaches the edge of the sphere of influence of the earth, it is then considered to instantaneously be under the sole affect of the sun. It must be emphasized that this is only an approximation to be used in the initial design of orbital transfers, with numerical techniques being employed to refine the initial orbit layout.

Now that the transfer time is determined, and everyone is happy at headquarters with the time schedule, it is time to look closer at the requirements to fling the satellite free of Earth's gravitational pull.

Earth Departure

Fig. 3 illustrates the satellite's departure from Earth. Initially, the satellite may be in a circular orbit about the earth, but to send it on its way to Mars, it will be necessary to put the satellite on a hyperbolic trajectory. The velocity of the satellite at an infinite distance away is called the *hyperbolic excess speed*, which is (by definition) 0 for a parabolic orbit. This excess speed is, for all practical purposes, the same as the velocity of the satellite when it arrives at the end of Earth's sphere of influence, denoted by V_∞ in **Fig. 3**.

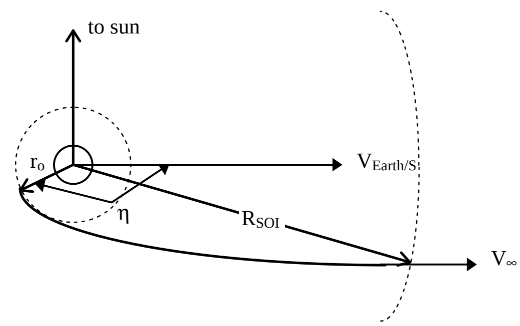

Fig. 3. Satellite Departure from Earth.

Since energy is constant along the hyperbolic escape orbit, we can equate the energy at *insertion* (the point where the engines are burned to place the spacecraft on the hyperbolic trajectory) and the energy at the point where the satellite departs from Earth's sphere of influence. If V_0 is the velocity of the initial circular orbit with radiud r_o, then

$$\varepsilon=\frac{V_0^2}{2}-\frac{\mu}{r_0}=\frac{V_\infty^2}{2}-\frac{\mu}{r_{SOI}}$$

which yields

$$V_0=\sqrt{V_\infty^2+2\frac{\mu}{r_0}-2\frac{\mu}{r_{SOI}}}$$

The last term under the radical is often neglected since it contributes little to the answer.

To get the maximum benefit of Earth's velocity, V_∞ should be parallel to V_{Earth} when the satellite crosses the edge of the sphere of influence. If this is the case, then the difference between the required velocity of the satellite relative to the sun ($V_{1/S}$ from **Fig. 2**) and the velocity of the earth ($V_{Earth/S}$ in **Fig. 3**) must be the required V_∞. Knowing this allows the calculation of V_0. This velocity is relative to the earth. Assuming that the satellite is originally in a circular orbit, the initial ΔV required to place the satellite into this hyperbolic trajectory is easily determined.

The final problem in the escape orbit is to determine where in the circular orbit to apply the ΔV. In other words, what is the angle η? Without going into the derivation, this angle can be calculated as follows:

1. Determine h from

$$h = r_0 V_0$$

2. Calculate the eccentricity of the hyperbolic orbit (noting that this is the eccentricity of the orbit relative to the earth, and not the eccentricity of the heliocentric transfer orbit):

$$e = \sqrt{1 + 2\varepsilon h^2 / \mu^2}$$

where the energy is that of the hyperbolic orbit.

3. Determine η by

$$\cos(\eta) = -\frac{1}{e}$$

Arrival at the Target Planet

The final, and most complicated, step of the interplanetary transfer is arrival at the target planet. The satellite may enter the sphere of influence of the target at a variety of relative angles and speeds, all of which must be considered.

If the satellite is launched when the phase angle is exactly the calculated value θ, then both the satellite and the planet will be in the same place at the same time (theoretically). However, we know that when the satellite enters the sphere of influence of the target planet, which for our example is Mars, then its orbit ceases to be an elliptical one dominated by the sun, and begins to be a hyperbolic orbit about Mars.

We will examine one and only one case, which is where the satellite enters the Martian sphere of influence at precisely the moment it crosses the orbit of Mars. With this simplification, calculation of the flight path angle and velocity relative to Mars allows all else to be determined. This special case is illustrated in **Fig. 4**. First, the flight path angle ϕ can be calculated since the angular momentum of the heliocentric transfer orbit is constant. Angular momentum can be evaluated back where the satellite left the orbit of Earth:

$$h_t = r_{Earth} V_{1/S}$$

Relative to the heliocentric orbit, the angular momentum at the point where the satellite enters the sphere of influence of Mars is also h_t, so

$$\cos(\phi) = h_t / r_f V_{2/S}$$

where r_f is the orbital radius of Mars, and $V_{2/S}$ is the velocity of the satellite relative to the sun at the point where it crosses Mars' orbit (see **Fig. 2**). This allows the calculation of V_∞, the velocity of the spacecraft relative to Mars when it crosses into the Martian sphere of influence. This vector addition is diagrammed in **Fig. 4** which yields the equation:

$$V_\infty^2 = V_{2/S}^2 + V_{Mars}^2 - 2V_{2/S}V_{Mars}\cos(\phi)$$

The angle θ in the figure can be determined from the law of sines:

$$\sin(\theta) = \frac{V_{2/S}}{V_\infty}\sin(\phi)$$

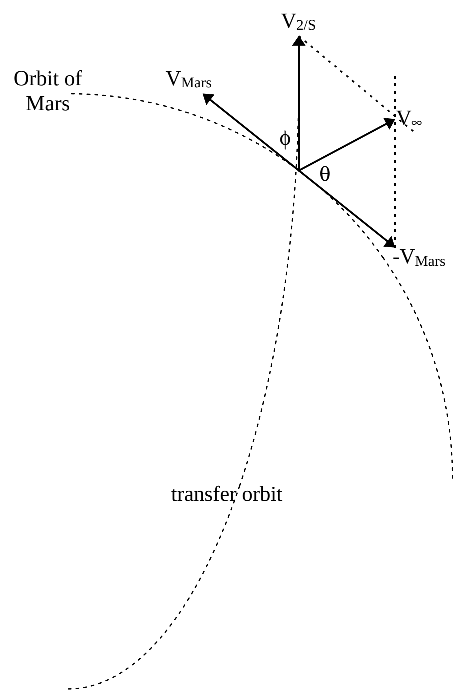

Fig. 4. The Satellite's Relative Velocity when it Encounters the Sphere of Influence of Mars.

Now we change reference frames and look at the orbit relative to Mars. The angle θ is related to the flight path angle of the satellite relative to Mars, and since V_∞ is also known, the shape and size of the orbit is fixed since energy and angular momentum can be calculated. Just how close does the satellite get to the Martian surface? Both r_p and V_p (radius and velocity at periapsis) are unknown. However, since the angular momentum is constant, and the flight path angle is 0 at periapsis,

$$V_p = \frac{h}{r_p}.$$

Since the energy is constant, the velocity can also be written as

$$V_p = \sqrt{V_\infty^2 + 2\frac{\mu}{r_p} - 2\frac{\mu}{r_{SOI}}}$$

where r_{SOI} is the radius of the sphere of influence of Mars. Thus we have two equations and two unknowns. Once r_p is determined, the ΔV required to slow the satellite enough to enter a circular orbit can be found.